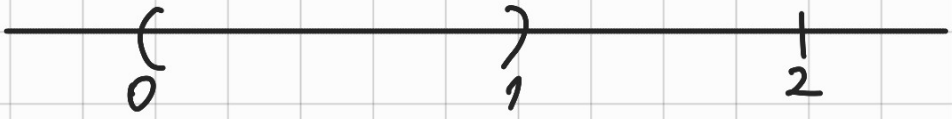
1. Perform the topological analysis of the set $A \subset \mathbb{R}$, where:

a) $A = (0,1) \cup \{2\}$

b) $A = \mathbb{N}$

c) $A = \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$

Sol a)



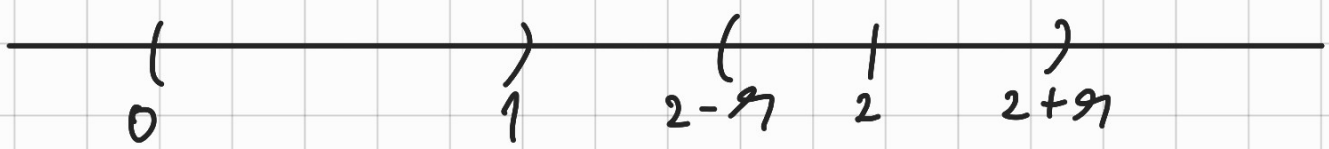
1) $(0,1)$ open set, $(0,1) \subset A$ $\Rightarrow (0,1) \subset \mathring{A}$

$\mathring{A} \subset A$

Hence $(0,1) \subset \mathring{A} \subset (0,1) \cup \{2\}$

We study if $2 \in \mathring{A}$

$2 \in \mathring{A} \Leftrightarrow \exists\, r > 0$ s.t. $(2-r; 2+r) \subset A$



Hence $2 \notin \mathring{A}$

Therefore $\mathring{A} = (0,1)$

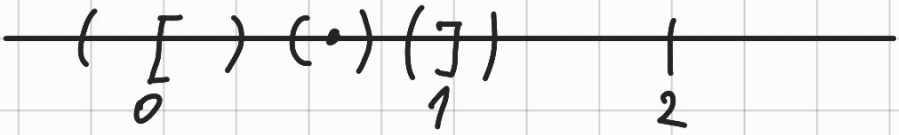
2) $\bar{A} = [0,1] \cup \{2\}$

3) $A' = ?$

$A' \subset \bar{A} = [0,1] \cup \{2\}$

$x \in A' \Leftrightarrow \forall\, r > 0,\ (x-r, x+r) \cap (A \setminus \{x\}) \neq \phi$

Let $x \in [0,1]$

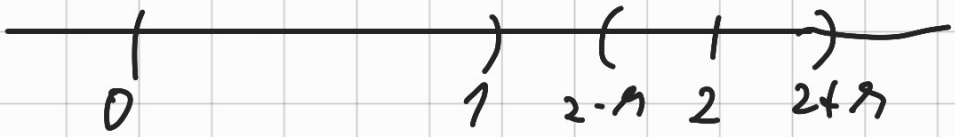


Hence $x \in A'$, i.e. $[0,1] \subset A'$

We will check if $2 \in A'$

$2 \in A' \Leftrightarrow \forall\, r > 0\ (2-r, 2+r) \cap (A \setminus \{2\}) \neq \phi$

$\Leftrightarrow (2-r, 2+r) \cap (0,1) \neq \phi$



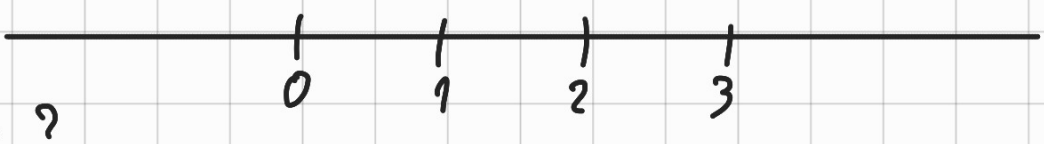
Hence, $2 \notin A'$

So $A' = [0, 1]$

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \{0, 1, 2\}$

5) $Iso(A) = {}'A = \bar{A} \setminus A' = \{2\}$ ▨

Sol b) $A = \mathbb{N}$



1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists \eta > 0$ s.t. $(x-\eta, x+\eta) \subset A$

$\mathring{A} = \phi$ because between any two real numbers there exits infinitely many rational numbers and infinitely

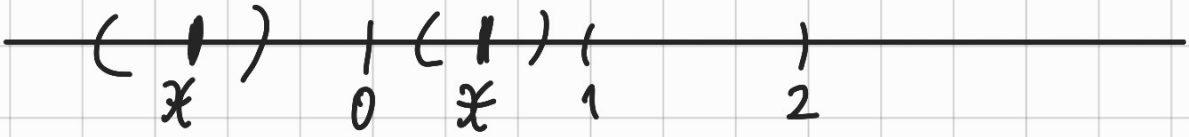
many irrational numbers.

2) $A' = ?$

$x \in A' \Leftrightarrow \forall \eta > 0$ we have

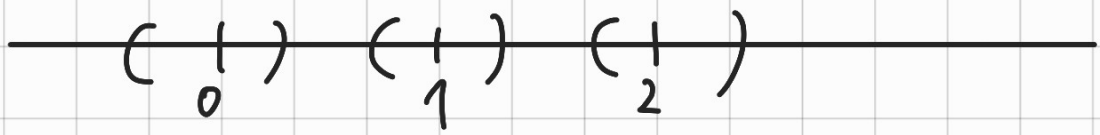
$(x-\eta, x+\eta) \cap (A \setminus \{x\}) \neq \phi$

Let $x \in \mathbb{R} \setminus A$



Hence $x \notin A'$

Let $x \in A$



Hence $x \notin A'$

Therefore $A' = \phi$

3) $\overline{A} = A \cup A' = \mathbb{N}$

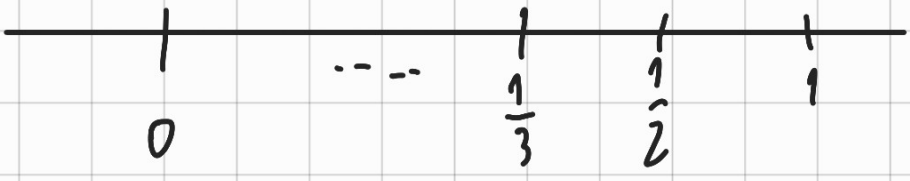
4) $Fr(A) = \partial A = \overline{A} \setminus \mathring{A} = \mathbb{N}$

5) $Iso(A) = {}'A = \tilde{A} \setminus A' = \mathbb{N}$ ▨

Ex c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \}$

1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists\, r > 0$ s.t. $(x-r, x+r) \subset A$



$\mathring{A} = \emptyset$ because any two real numbers, there exists infinitly many rational numbers and infintely many irrational numbers.

2) $A' = ?$

$x \in A' \Leftrightarrow \exists (x_k) \subset A \setminus \{x\}$ s.t.

$\lim_{k \to \infty} x_k = x$

Let $(x_k)_k \subset A$ s.t. $(x_k)_k$ is conv.

Hence $\lim_{k \to \infty} x_k$ could be equal to 0 $\left(\frac{1}{k} \xrightarrow[k \to \infty]{} 0\right)$ or to every element in A (the constant sequences from a rank)

Hence $A' = \{0\}$

3) $\bar{A} = A \cup A' = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\} \cup \{0\}$

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\} \cup \{0\}$

5) $Iso(A) = {}'A = \bar{A} \setminus A' = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$

We consider the metric space

$(\mathbb{R}^2, \underset{\text{not } d}{d_2})$, where $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$d((x,y);(z,t)) = \sqrt{(x-z)^2+(y-t)^2}$$

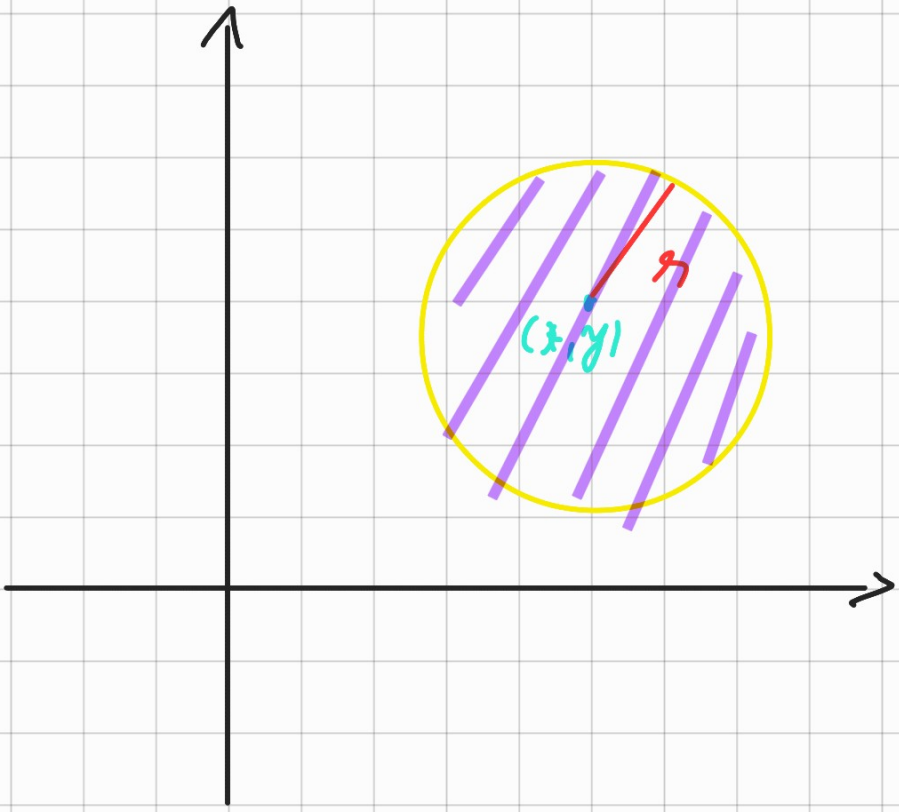
Let $(x,y) \in \mathbb{R}^2$ and $r > 0$

1) $B((x,y), r) = \{(z,t) \in \mathbb{R}^2 \mid \sqrt{(x-z)^2+(y-t)^2} < r\}$

$= \{(z,t) \in \mathbb{R}^2 \mid (x-z)^2+(y-t)^2 < r^2\}$

$= \{(z,t) \in \mathbb{R}^2 \mid (z-x)^2+(t-y)^2 < r^2\}$

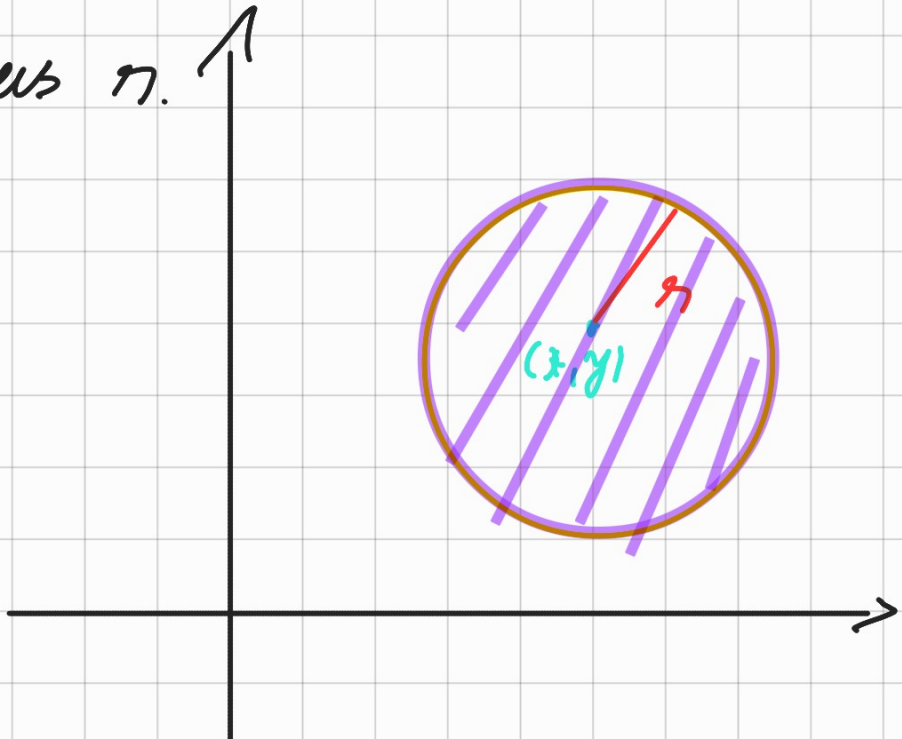
= the open disk with center (x,y) and radius r



2) $B[(x,y), r] = \overline{B}((x,y), r) =$

$= \{(z,t) \in \mathbb{R}^2 \mid (z-x)^2 + (t-y)^2 \leq r^2\} =$

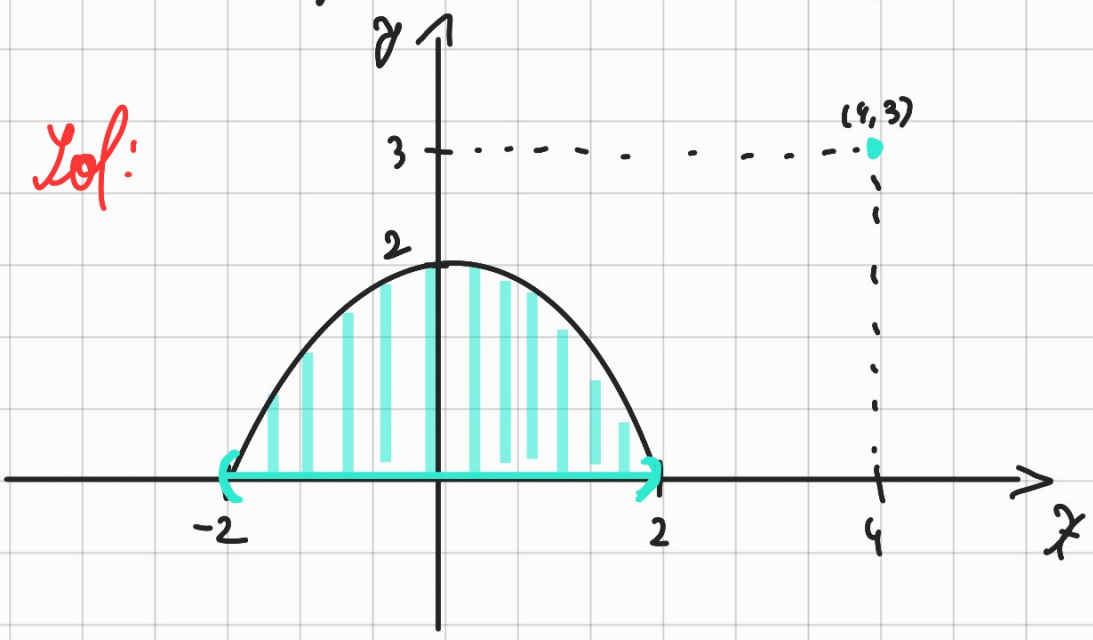
= closed disk with center (x,y) and radious r.



2) Perform the Topological analysis of $A \subset \mathbb{R}^2$, where

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 4,\ y \geq 0\} \cup \{(4,3)\}$$

Sol:



a) $\mathring{A} = ?$

$(x,y) \in \mathring{A} \iff \exists\, r > 0$ s.t. $B((x,y),r) \subset A$

$\mathring{A} \subset A$

$\{(x,y) \mid x^2+y^2 < 4,\ y > 0\}$ open set

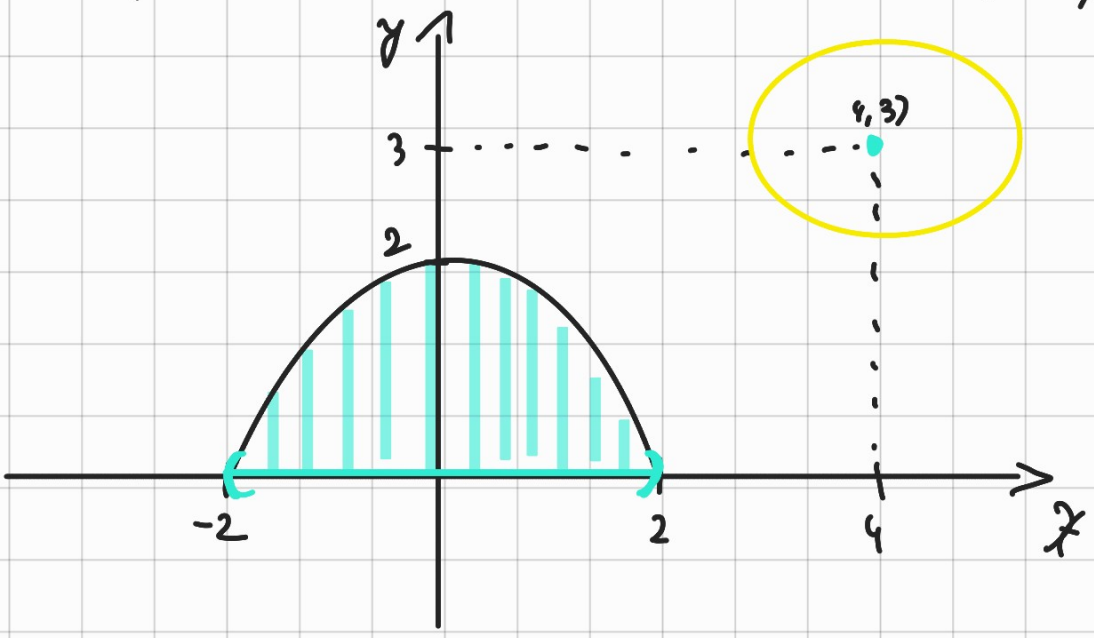
$\{(x,y) \mid x^2+y^2 < 4,\ y > 0\} \subset A$

Hence $\{(x,y) \mid x^2+y^2<4, y>0\} \subset \mathring{A} \subset$

$\subset \{(x,y) \mid x^2+y^2<4, y\geq 0\} \cup \{(3,4)\}$

We check if $\{(x,0) \in \mathbb{R}^2 \mid x^2<4\} \cup \{(3,4)\} \subset \mathring{A}$

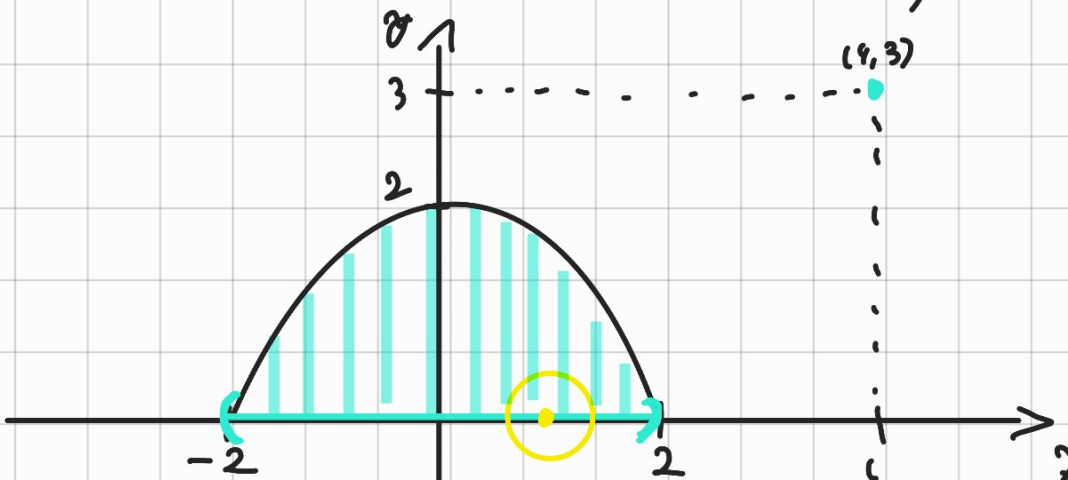
$(4,3) \in \mathring{A} \Leftrightarrow \exists\, r>0$ s.t. $B((4,3),r) \subset A$



Hence $(4,3) \notin \mathring{A}$

Let $x \in \mathbb{R}$; $x^2<4$

$(x,0) \in \mathring{A} \Leftrightarrow \exists\, r>0$ s.t. $B((x,0),r)$



Hence $(x, 0) \notin \mathring{A}$

Therefore $\mathring{A} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4, y > 0\}$

b) $\bar{A} = ?$

$(x, y) \in \bar{A} \Leftrightarrow \forall r > 0$, we have $B((x, y), r) \cap A \neq \phi$

$\{(x, y) \mid x^2 + y^2 \leq 4 ; y \geq 0\} \cup \{(4, 3)\}$ is a closed set $\Rightarrow$

$A \subset \{(x, y) \mid x^2 + y^2 \leq 4; y \geq 0\} \cup \{(4, 3)\}$

$\Rightarrow \bar{A} \subset \{(x, y) \mid x^2 + y^2 \leq 4; y \geq 0\} \cup \{(4, 3)\}$

$A \subset \bar{A}$

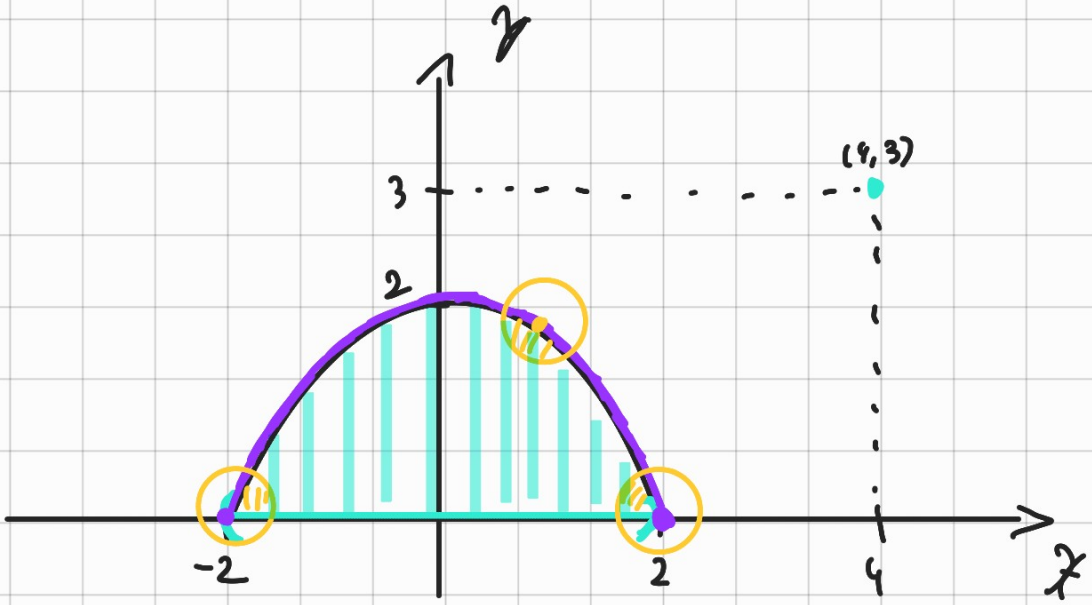
Hence, $\{(x, y) \mid x^2 + y^2 < 4; y \geq 0\} \cup \{(4, 3)\} \subset \bar{A} \subset$

$\subset \{(x, y) \mid x^2 + y^2 \leq 4; y \geq 0\}$

We check if $\{(x,y) \mid x^2+y^2=4, y\geq 0\} \subset \bar{A}$

Let $(x,y)\in\mathbb{R}^2$ s.t. $x^2+y^2=4$; $y\geq 0$

$(x,y)\in \bar{A} \Leftrightarrow \forall \eta > 0$ we have $B((x,y),\eta)\cap A\neq\phi$



Hence $(x,y)\in\bar{A}$

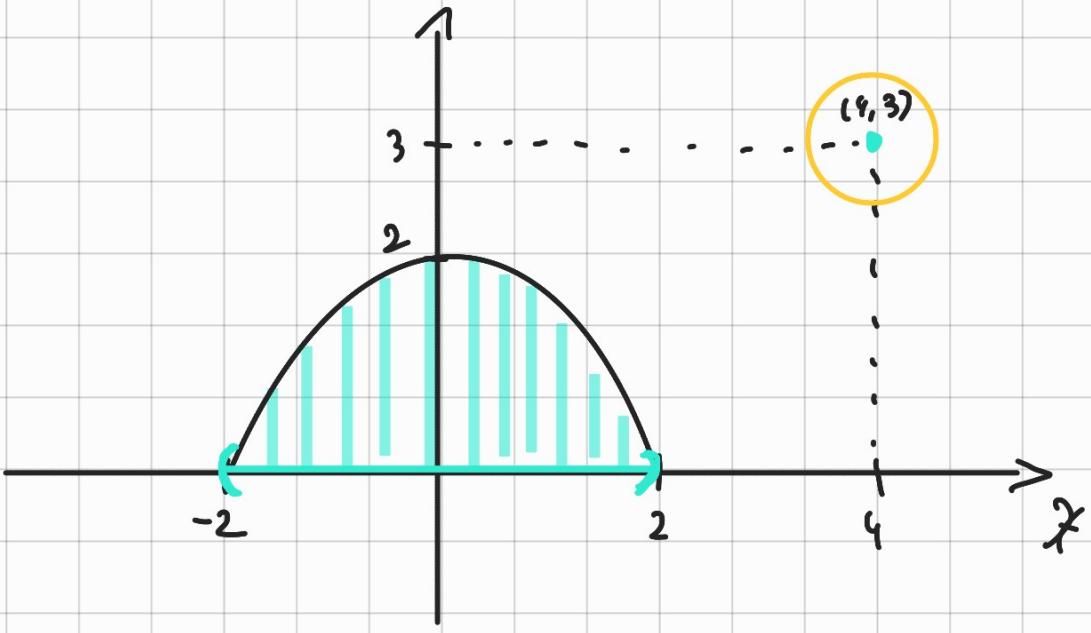
So $\bar{A}=\{(x,y) \mid x^2+y^2\leq 4, y\geq 0\}$

c) $A' = ?$

$(x,y)\in A' \Leftrightarrow \forall \eta>0, B((x,y),\eta)\cap A\setminus\{(x,y)\}\neq\phi$

$A'\subset\bar{A}=\{(x,y)\in\mathbb{R} \mid x^2+y^2\leq 4, y\geq 0\}\cup\{(4,3)\}$

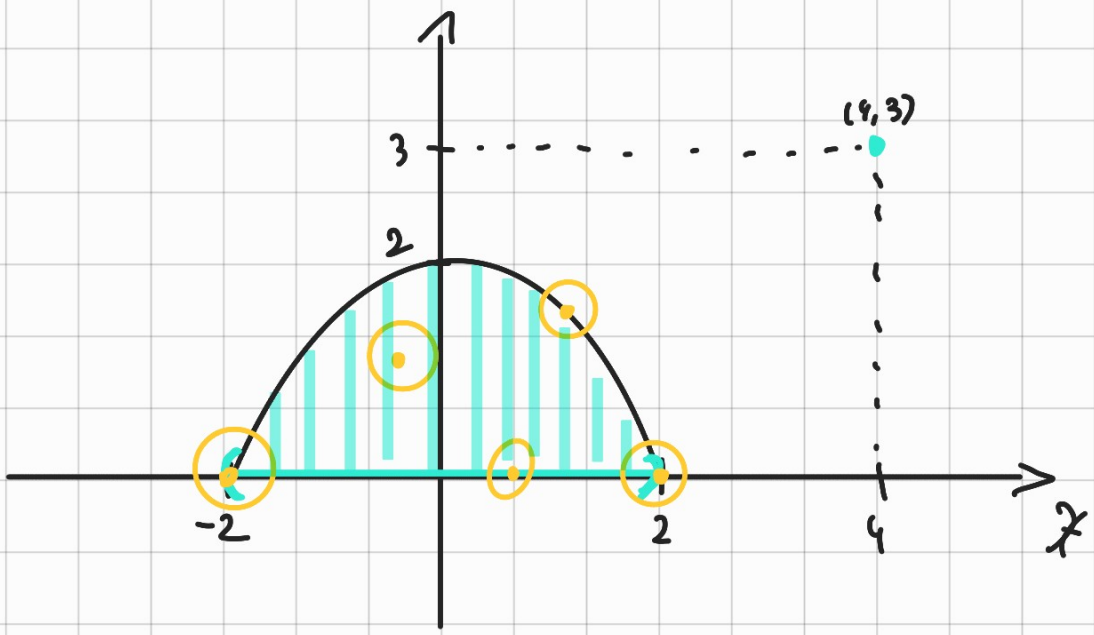
$(4,3) \in A' \iff \forall r > 0,\ B((4,3), r) \cap (A \setminus (4,3)) \neq \emptyset$



Hence $(4,3) \notin A'$

We check if $\{(x;y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4; y \geq 0\} \subset A'$

Let $(x,y) \in \mathbb{R}^2$ s.t. $x^2+y^2 \leq 4,\ y \geq 0$



Hence $(x,y) \subset A'$

Therefore $\{(x,y) \mid x^2+y^2 \leq 4 ; y \geq 0\} \subset A'$

So, $A' = \{(x,y) \mid x^2+y^2 \leq 4 ; y \geq 0\}$

d) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} =$

$= [\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq, y \geq 0\} \cup \{(4,3)\} \setminus$

$\setminus \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 4, y \geq 0\} \cup \{(4,3)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 4, y \geq 0\} \cup$

$\cup \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 4, y = 0\} \cup$

$\cup \{(4,3)\}$

e) $Isol(A) = A = \bar{A} \setminus A' = \{(4,3)\}$ ☒