1. Study the convergence (nature) of the following series:

a) $\sum_{x=1}^{\infty} \frac{1}{n\sqrt{n+1}} x^n, \; x>0$

Sol a) Let $x_n = \frac{1}{n\sqrt{n+1}} x^n, \; \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(\frac{1}{(n+1)\cdot\sqrt{n+2}} \cdot x^{n+1} \right) \cdot \left(n\cdot\sqrt{n+1} \cdot \frac{1}{x^n} \right)$$

$$= \lim_{n\to\infty} \frac{n\cdot\sqrt{n+1}}{(n+1)\cdot\sqrt{n+2}} \cdot x =$$

$$= x \lim_{n\to\infty} \frac{n}{n+1} \cdot \sqrt{\frac{n+1}{n+2}} = x$$

According to Ratio criterion we have:

1) If $x<1$ (i.e. $x \in (0,1)$), then the $\sum_{n=1}^{\infty} x_n$ is conv.

2) If $x > 1$ (i.e. $x \in (1, \infty)$), then $\sum_{n=1}^{\infty} x_n$ is div

3) If $x = 1$ the criterion does not decide

Let $x = 1$

$$x_n = \frac{1}{n\sqrt{n+1}} \cdot 1^n = \frac{1}{n\sqrt{n+1}} \quad \forall\, n \in \mathbb{N}^*$$

Let $y_n = \frac{1}{n\sqrt{n}} \quad \forall\, n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n\sqrt{n}}{n\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 \in (0, +\infty)$$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \quad \text{conv. (gen. h. s., } \alpha = \tfrac{3}{2})$$

According to the Comparison criterion with limits, $\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$.

Hence, $\sum_{n=1}^{\infty} x_n$ is also conv. ▨

b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in (-1, 1)$

$x_n = \frac{x^n}{n^2}$, $\forall\, n \in \mathbb{N}^*$

$|x_n| = \left|\frac{x^n}{n^2}\right| = \frac{|x^n|}{|n^2|} = \frac{|x|^n}{n^2} < \frac{1}{n^2}$ $\forall\, n \in \mathbb{N}^*$

$\sum_n \frac{1}{n^2}$ is convergent (g. h. s. $\alpha = 2$)

By Comparison criterion with inequalities, $\sum_n |x_n|$ is. conv. Hence $\sum_n x_n$ is.

absolutely conv., so $\sum_n x_n$ is conv.



c) $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{\lambda}}$, $x \in \mathbb{R}$, $\lambda > 0$

Let $a_n = \frac{1}{n^{\lambda}}$, $\forall\, n \in \mathbb{N}^*$

$b_n = \cos(nx)$

We will apply Abel-Dirichlet criterion (I)

$(a_n)_n$ is (strictly) decreasing and

$\lim_{n \to \infty} a_n = 0$ (1)

We look for $M > 0$ s.t. $\forall\, n \in \mathbb{N}^*$, we have $|b_1 + b_2 + \dots + b_n| \leq M$

(M should not depend on n, but it could depend on x)

Let $z = \cos x + i \sin x$

$z^2 = (\cos x + i \sin x)^2 = \cos(2x) + i \sin(2x)$

.....

$z^n = (\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

$Re\,(z + z^2 + \dots + z^n) = b_1 + b_2 + \dots + b_n \quad \forall n \in \mathbb{N}^*$

Let $n \in \mathbb{N}^*$. We assume $z \neq 1$,

i.e. $\cos x + i \sin x \neq 1$,

i.e. $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

$$z + z^2 + \dots + z^n = \frac{z^{n+1} - z}{z - 1} =$$

$$= \frac{\cos[(n+1)x] + i \sin[(n+1)x] - (\cos x + i \sin x)}{\cos x + i \sin x - 1}$$

$$= \frac{\cos[(n+1)x] - \cos x + i\{\sin[(n+1)x] - \sin x\}}{(\cos x - 1) + i \sin x} =$$

$$= \frac{-2 \sin\left[\frac{(n+2)x}{2}\right] \sin\left(\frac{nx}{2}\right) + i2\left\{\cos\left[\frac{(n+2)x}{2}\right] \sin\left(\frac{nx}{2}\right)\right\}}{-2 \sin^2\left(\frac{x}{2}\right) + i2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)}$$

$-i$

$$= \frac{\sin\left(\frac{nx}{2}\right)\left(\cos\frac{n+2}{2}x + i \sin\frac{n+2}{2}x\right)}{\sin\frac{x}{2}\left(\cos\frac{x}{2} + i \sin\frac{x}{2}\right)} =$$

$$= \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}} \cdot \frac{\left(\cos\frac{x}{2} + i \sin\frac{x}{2}\right)^{n+2}}{\cos\frac{x}{2} + i \sin\frac{x}{2}}$$

$n+2 \to n+1$

$$= \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \cdot \left(\cos \frac{(n+1)x}{2} + i \sin \frac{(n+1)x}{2} \right)$$

Hence $\mathrm{Re}(z + z^2 + \dots + z^n) =$

$$= \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \cdot \cos \frac{(n+1)x}{2}$$

$$|b_1 + \dots + b_n| = |\mathrm{Re}(z + z^2 + \dots z^n)| =$$

$$= \left| \frac{\sin \frac{nx}{2} \cos \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \right| =$$

$$= \frac{\left| \sin \frac{nx}{2} \right| \cdot \left| \cos \frac{(n+1)x}{2} \right|}{\left| \sin \frac{x}{2} \right|} \leq \frac{1 \cdot 1}{\left| \sin \frac{x}{2} \right|} = \frac{1}{\left| \sin \frac{x}{2} \right|}$$

We choose $M = \frac{1}{\left|\sin \frac{x}{2}\right|}$

According to Abel-Dirichlet criterion (I) we have that

$\sum_{n=1}^{\infty} a_n \cdot b_n$ is conv $(\forall \ \lambda > 0)$

But the proof from above was with $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

Let $x \in \{2k\pi \mid k \in \mathbb{Z}\}$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}} = \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}}$$

conv., if $\lambda \in (1, \infty)$

div., if $\lambda \in (0, 1]$



d) $\sum_{n=1}^{\infty} \frac{\cos n \cdot \cos \frac{1}{n}}{n}$

Sol: Let $a_n = \cos \frac{1}{n}$, $\forall\, n \in \mathbb{N}^*$

$b_n = \frac{\cos n}{n}$

$-1 \leq \cos \frac{1}{n} \leq 1 \quad \forall\, n \in \mathbb{N}^* \Rightarrow$

$\Rightarrow (a_n)_n$ is bounded

$x \longmapsto \cos x$ is strictly decreasing

$\cap$ $\quad\quad\quad

Hence $(a_n)_n$ is bounded and (strictly) monotonic (1)

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\cos n}{n} = \sum_{n=1}^{\infty} \frac{\cos n \cdot 1}{n^1} \text{ conv} \quad (2)$$

(see ex c) with $x=1, \lambda=1$)

By (1) + (2), according to Abel-Dirichlet criterion (II) we have that

$$\sum_{n=1}^{\infty} a_n \cdot b_n \text{ is conv.} \quad \blacksquare$$

l) $$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n} + 1}{n}$$

$$x_n = \frac{(-1)^n \sqrt{n} + 1}{n} \; ; \forall n \in \mathbb{N}^*$$

$$x_n = (-1)^n \cdot \frac{\sqrt{n}}{n} + \frac{1}{n} \; ; \forall n \in \mathbb{N}^*$$

Let $y_n = (-1) \cdot \frac{\sqrt{n}}{n}$; $\forall n \in \mathbb{N}^*$ s.t. $x_n = y_n + z_n$

$z_n = \frac{1}{n}$

$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (gen. R.S., $\alpha = 1$)

Let $a_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ $\forall n \in \mathbb{N}^*$ s.t.

$y_n = (-1)^n \cdot a_n \quad \forall n \in \mathbb{N}^*$

$(a_n)_n \subset [0, \infty)$ (strictly) decreasing and $\lim_{n \to \infty} a_n = 0$

By Leibniz criterion it follows that

$\sum_{n=1}^{\infty} y_n$ is conv.

Therefore, $\sum_{n=1}^{\infty} x_n$ is div.

2. Let $n \in \mathbb{N}^*$ and $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = |x_1 - y_1| + \dots + |x_n - y_n|$$

where $x = \{x_1, x_2, \dots, x_n\}$

$y = \{y_1, y_2, \dots, y_n\}$

Prove that $(\mathbb{R}, d_1)$ is a metric space.

Let $X, Y, Z \in \mathbb{R}^n$

1) $d_1(X,Y) = \sum_{i=1}^{n} |x_i - y_i| \geq 0$

2) $d_1(X,Y) = 0 \iff \sum_{i=1}^{n} |x_i - y_i| = 0 \iff$

$\iff |x_i - y_i| = 0 \quad , \forall i \in \overline{1,n}$

$\iff x_i - y_i = 0 \quad , \forall i \in \overline{1,n}$

$\iff x_i = y_i \quad , \forall i \in \overline{1,n} \iff X = Y$

3) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(y, x)$

4) $d_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq$

$\leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) =$

$= \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d(x, y) + d_1(y, z)$

Hence $(\mathbb{R}^n, d_1)$ is a metric space ▨