Theorem Let $\phi \neq A \subset \mathbb{R}$, $a \in A \cap A'$, $f: A \to B$ be s.t. f is diff (differentiable) at a, where $\phi \neq B \subset \mathbb{R}$. Let $g: B \to \mathbb{R}$. We assume that $f(a) \in B \cap B'$ and that g is diff. in $f(a)$. Then $g \circ f: A \to \mathbb{R}$ is diff. at a and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Def: Let $\phi \neq A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ and $a \in A$. We say that a is a:

1) local minimum point of A if
$\exists V \in \mathcal{V}_a$ s.t. $f(a) \leq f(x) \quad \forall x \in V \cap A$

2) local maximum point of f if
$\exists V \in \mathcal{V}_a$ s.t. $f(a) \geq f(x) \quad \forall x \in V \cap A$

3) local extremum point of f if a is a local minimal or maximum point of f.

Theorem (Fermat Theorem) Let $\phi \neq A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ and $a \in A$ s.t.:

1) $a \in \mathring{A}$
2) a is a local extremum point of f.
3) f is diff. at a

Then $f'(a) = 0$

Theorem (Rolle Theorem). Let $a, b \in \mathbb{R}$, $a < b$ and $f: [a, b] \to \mathbb{R}$ s.t.

1) f is cont. on $[a, b]$
2) f is diff on (a, b)
3) $f(a) = f(b)$

Then $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$

Theorem (Lagrange Theorem). Let $a, b \in \mathbb{R}$ $a < b$ and $f: [a, b] \to \mathbb{R}$

1) f is cont. on $[a, b]$
2) f is diff on (a, b)

Then $\exists\, c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Corollary Let $I \subset \mathbb{R}$ be a nondegenerated interval (i.e. $I \neq \emptyset$ and I is not reduced to a single point) and $f: I \to \mathbb{R}$ a diff. function (on I). Then:

1) $(f'(x) = 0 \quad \forall x \in I) \Leftrightarrow (f$ is constant (on I))

2) $(f'(x) \geq 0 \quad \forall x \in I) \Leftrightarrow (f$ is increasing on I))

2') $(f'(x) > 0 \quad \forall x \in I) \Rightarrow (f$ is strictly increasing (on I))

3) $(f'(x) \leq 0 \quad \forall x \in I) \Leftrightarrow (f$ is decreasing (on I))

3') $(f'(x) < 0 \quad \forall x \in I) \Rightarrow (f$ is strictly decreasing (on I))

Corollary Let $I \subset \mathbb{R}$ be a nondegenerated interval, $a \in I$ and $f: I \to \mathbb{R}$ s.t.

1) f is cont. on I
2) f is diff. on $I \setminus \{a\}$
3) $\exists \lim_{x \to a} f'(x) \overset{\text{not.}}{=} \ell \in \mathbb{R} \cup \{\pm \infty\}$

Then f has a derivative at a and $f'(a) = \ell$

Theorem (Cauchy Theorem). Let $a, b \in \mathbb{R}$ $a < b$ and $f, g: [a,b] \to \mathbb{R}$ be s.t.:

1) f, g cont. on $[a,b]$
2) f, g diff. on (a,b)

Then $\exists\ c \in (a,b)$ s.t.

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

Theorem (Darboux Theorem). Let $I \subset \mathbb{R}$ be a nondegenerated interval and $f: I \to \mathbb{R}$ s.t. f is diff. (on I). Then, for any interval $J \subset I$, we have that $f(J)$ is an interval

Theorem (L'Hospital Theorem)

Let $a, b \in \overline{\mathbb{R}}$, $a < b$ and $I \subset \mathbb{R}$ be an interval s.t. $(a,b) \subset I \subset [a,b]$, $x_0 \in I$ and $f, g: I \setminus \{x_0\} \to \mathbb{R}$ with the following properties:

1) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$

(resp. $\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = \infty$)

2) f and g are diff (on $I \setminus \{x_0\}$ and $g'(x) \neq 0 \quad \forall x \in I \setminus \{x_0\}$

3) $\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \overset{not}{=} l \in \overline{\mathbb{R}} \overset{not.}{=} \mathbb{R} \cup \{\pm\infty\}$

Then:

i) $g(x) \neq 0 \quad \forall x \in I \setminus \{x_0\}$

(resp $\exists V \in \mathcal{V}_{x_0}$ s.t. $g(x) \neq 0$

$\forall x \in V \cap (I \setminus \{x_0\})$)

ii) $\exists \lim_{x \to x_0} \frac{f(x)}{g(x)}$ and $\lim_{x \to x_0} \frac{f(x)}{g(x)} = l$

Let $\emptyset \neq A \subset \mathbb{R}$, $a \in A \cap A'$ and $f: A \to \mathbb{R}$

Def: We say that f is twice differentiable at a if $\exists\, V \in \mathcal{V}_a$ s.t. f is diff. on $V \cap A$ and $f': V \cap A \to \mathbb{R}$ is differentiable at a. The derivative of $f'(a)$ is called the second derivative of f at a or the derivative of second order at a and it is denoted by $f''(a)$ or $f^{(2)}(a)$

Def: If f is twice differentiable on a non-void set $\emptyset \neq B \subset A$, we can define the function

$$f'': B \to \mathbb{R}, \quad \begin{array}{ccc} x & \longmapsto & f''(x) \\ \uparrow & & \uparrow \\ B & & \mathbb{R} \end{array}$$

and we call this function the second derivative of f or the derivative of second order

Obs: Similarly we can define the n-th derivative of f, denoted by $f^{(n)}$ $\forall n \in \mathbb{N}$, $n \geq 3$

Let $I \subset \mathbb{R}$ be a nondegenerated interval, $n \in \mathbb{N}$ and $f: I \to \mathbb{R}$

Def: We say that f is of class C^n (on I) if f is n times differentiable (on I) and $f^{(n)}: I \to \mathbb{R}$ is continuous (on I)

Notation

$C^n(I) = \{ g: I \to \mathbb{R} \mid g \text{ is of class } C^n \text{ (on } I) \}$

$C^0(I) \overset{not}{=} C(I) = \{ g: I \to \mathbb{R} \mid g \text{ is cont. (on } I) \}$

Def: We say that f is of class C^∞ (on I) if f is infinitely many times differentiable

Notation

$C^\infty(I) = \{g : I \to \mathbb{R} \mid g \text{ is of class } C^\infty \text{ (on } I)\}$

Theorem (Taylor Formula with Lagrange remainder)

Let $I \subset \mathbb{R}$ be a nondegenerated interval, $a \in I$, $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ s.t. f is $n+1$ times diff. (on I). Then, $\forall x \in I$, $x \neq a$, $\exists c$ between a and x (i.e. $c \in (a, x)$ or $c \in (x, a)$) s.t.

$$f(x)=\underbrace{f(a)+\frac{f'(a)}{1!}(x-a)+\dots+\frac{f^{(n)}(a)}{n!}\cdot(x-a)^n}_{\text{not } T_n(x)}+\underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}}_{\text{not } R_n(x)}$$

$T_n(x)$ (Taylor polynomial of order n)

$R_n(x)$ (Lagrange reminder of order n)

Obs: $T_n(x)$, $R_n(x)$ and c from the preceding theorem depend on n, x, a and f

Def: Let $I \subset \mathbb{R}$ be a nondegenerated interval and $f, F : I \to \mathbb{R}$ s.t. F is differentiable (on I) and $F'(x) = f(x) \quad \forall x \in I$. We say that I is an antiderivative of f.

Uniformly continuous functions

Def: Let (X, d_1) and (Y, d_2) be metric spaces, $\emptyset \neq A \subset X$ and $f: A \to Y$. We say that f is uniformly continuous (on A) if $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. $\forall x, y \in A$ with the property that $d_1(x, y) < \delta_\varepsilon$, we have that $d_2(f(x), f(y)) < \varepsilon$

Prop In the context of the preceding definition, if f is uniformly continuous (u.c.) (on A), it follows that f is continuous (on A).

Obs The reciprocal of the previous proposition is false.

Prop In the context of the preceding def., if A is compact and if f is cont. (on A), then f is u.c. (on A)

Prop. Let (X, d_1) ; (Y, d_2) be metric spaces, $\emptyset \neq A \subset X$ and $f: A \to Y$.

The following are equivalent:

1) f is u.c. (on A)

2) $\forall (x_n)_n \subset A$, $\forall (y_n)_n \subset A$ s.t.

$\lim_{n\to\infty} d_1(x_n, y_n) = 0$, we have

$\lim_{n\to\infty} d_2(x_n, y_n) = 0$

Prop: Let $\emptyset \neq A \subset \mathbb{R}$ and $f: A \to \mathbb{R}$. The following are equivalent:

1) f is u.c. (on A)

2) $\forall (x_n)_n \subset A$, $\forall (y_n)_n \subset A$, s.t. $\lim_{n\to\infty} (x_n - y_n) = 0$,

we have that $\lim_{n\to\infty} (f(x_n) - f(y_n)) = 0$

Prop: Let $I \subset \mathbb{R}$ be a nondegenerated interval and $f: I \to \mathbb{R}$ a differentiable function (on I) s.t. the derivative of f is bounded (i.e. $\exists a, b \in \mathbb{R}$ s.t. $a \leq f'(x) \leq b \quad \forall x \in I$, i.e. $\exists M > 0$ s.t. $|f'(x)| \leq M \quad \forall x \in I$). The f is u.c.

Prop: Let $\emptyset \neq A \subset \mathbb{R}$ (A is not necessarily an interval) and $f: A \to \mathbb{R}$. The following are equivalent:

1) f is u.c. (on A)
2) $\exists\, a \in A$ s.t. f is u.c. on $A \cap (-\infty, a]$ (i.e. $f|_{A\cap(-\infty,a]}$ is u.c.) and f is u.c. on $A \cap [a, \infty)$ (i.e. $f|_{A\cap[a,\infty)}$ is $[a,\infty)$)

Prop: Let $a, b \in \mathbb{R}$, $a < b$ and $f: (a,b] \to \mathbb{R}$. The following are equivalent:

1) f is u.c. (on $(a,b]$)
2) $\exists\ \tilde{f}: [a,b] \to \mathbb{R}$ cont. (on $[a,b]$) s.t. $\tilde{f}|_{(a,b]} = f$ ($\tilde{f}$ is a cont. extension of f)

Def: Let (X, d) be a metric space and $(x_n)_n \subset X$. We say that $(x_n)_n$ is a Cauchy sequence with respect to the metric d if $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall m, n \in \mathbb{N}$, $m \geq n_\varepsilon, n \geq n_\varepsilon$, we have $d(x_m, x_n) < \varepsilon$

Obs: The syntagm „with respect to the metric d" can be replaced by the syntagm „in the metric space (X, d)"

Prop: Let (X, d_1) and (Y, d_2) be metric spaces, $\emptyset \neq A \subset X$, $(x_n)_n \subset X$ a Cauchy sequence with respect to the metric d_1 and $f: A \to X$. Then

$(f(x_n))_n \subset Y$ is a Cauchy sequence with respect to the metric d_2.

Sequences of functions

Let $A \neq \phi$, (Y, d) be a metric space, $(f_n)_n$ a sequence of functions, $f_n : A \to Y \ \forall n \in \mathbb{N}$ and $f : A \to Y$

Def: We say that the sequence $(f_n)_n$ converges:

1) pointwise to f if, $\forall x \in A$, we have $\lim_{n \to \infty} f_n(x) \overset{d}{=} f(x)$

(i.e $\forall x \in A, \forall \varepsilon > 0, \exists n_{\varepsilon, x} \in \mathbb{N}$ s.t. $n \in \mathbb{N}, n \geq n_{\varepsilon, x}$, we have $d(f_n(x), f(x)) < \varepsilon$

2) uniformly to f if $\forall \varepsilon > 0$,
$\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall x \in A, \forall n \in \mathbb{N}$,
we have $d(f_n(x), f(x)) < \varepsilon$

Notations In the context of the preceding definition:

1) If $(f_n)_n$ converges to f we will denote by $f_n \xrightarrow[n\to\infty]{f} f$

2) If $(f_n)_n$ converges uniformly to f, we will denote by $f_n \xrightarrow[n\to\infty]{u} f$

Obs In the context of the preceding definition, if $f_n \xrightarrow[n\to\infty]{u} f$, then $f_n \xrightarrow[n\to\infty]{f} f$

Obs: The converse of the preceding observation is false.

Theorem In the context of the preceding definition, the following are equivalent:

1) $f_n \xrightarrow[n\to\infty]{u} f$

2) $\lim_{n\to\infty} \left(\sup_{x\in A} d(f_n(x), f(x)) \right) = 0$