Properties of the derived set

Let (X, τ) be a topological space and $A \subset X$

Note: A' = set of accumulation points

$\bar{A}$ = set of adherence points

1) $A' \subset \bar{A}$

Prof: Let $x \in A'$. Let $V \in \mathcal{V}_x$. We will show that $V \cap A \neq \phi$

$$\left.\begin{array}{l} x \in A' \\ V \in \mathcal{V}_x \end{array}\right\} \Rightarrow V \cap (A \setminus \{x\}) \neq \phi \Rightarrow$$

$$\Rightarrow V \cap A \neq \phi \Rightarrow$$

$$\Rightarrow x \in \bar{A}. \text{ Hence } A' \subset \bar{A}$$

2) $\bar{A} = A \cup A'$

„⊃"

$$\left.\begin{array}{l} \bar{A} \supset A \\ \bar{A} \supset A' \end{array}\right\} \Rightarrow \bar{A} \supset A \cup A'$$

„⊂"

Let $x \in \bar{A}$ We. will prove that $x \in A \cup A'$

If $x \in A$, then the proof is finished, because $x \in A \cup A'$

Let us assume that $x \notin A$. We will show that $x \in A'$. Let $V \in \mathcal{V}_x$

$$\left.\begin{array}{l} x \in \bar{A} \\ V \in \mathcal{V}_x \end{array}\right\} \Rightarrow V \cap \neq \phi$$

Because $x \notin A$ it follows that $V \cap (A \setminus \{x\}) \neq \phi$.

Therefore $x \in A'$. So $\bar{A} \subset A \cup A'$

Metric spaces

Def: Let $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}$ is called a metric (or a distance) on X if

1) $d(x,y) \geq 0 \quad \forall \quad x, y \in X$

2) $d(x,y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$

3) $d(x,y) = d(y,x) \quad \forall x, y \in X$

4) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$

(The triangle inequality)

Def: Let $X \neq \emptyset$ and d be a distance (or a metric) on X. The pair (X,d) is called a metric space.

Examples:

1) Let $X \neq \phi$ and $d: X \times X \to \mathbb{R}$,

$$d(x,y) = \begin{cases} 0 ; & x = y \\ 1 ; & x \neq y \end{cases}$$

The pair (X, d) is a metric space

2) Let $X = \mathbb{R}$ and $d: X \times X \to \mathbb{R}$

$$d(x,y) = |x - y|$$

The pair is (X, d) is a metric space

3) Let $n \in \mathbb{N}^*$, $X = \mathbb{R}^*$ and $d_1: X \times X \to \mathbb{R}$

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = |x_1 - y_1| + \ldots + |x_n - y_n|$$

$x = \{x_1, x_2, \ldots, x_n\}$
$y = \{y_1, y_2, \ldots, y_n\}$

The pair is (X, d_1) is a metric space

4) Let $n \in \mathbb{N}^*$, $X = \mathbb{R}^n$ and $d_2 : X \times X \to \mathbb{R}$,

$x = \{x_1, x_2, \dots, x_n\}$
$y = \{y_1, y_2, \dots, y_n\}$

$$d_2(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

The pair (X, d_2) is a metric space

5) Let $n \in \mathbb{N}^*$, $X = \mathbb{R}^n$ and $d_\infty : X \times X \to \mathbb{R}$,

$x = \{x_1, x_2, \dots, x_n\}$
$y = \{y_1, y_2, \dots, y_n\}$

$$d_\infty(x,y) = \max \{ |x_i - y_i| \mid i = \overline{1,n} \}$$

The pair (X, d_∞) is a metric space

Def: Let $n \in \mathbb{N}^*$ and the metric d_2 on $\mathbb{R}^n$. This metric is called the Euclidean metric on $\mathbb{R}^n$.

Remark: In the context of the preceding definition, when clear, we can write $d_2 \overset{not}{=} d$.

Def: Let (X, d) be a metric space, $x \in X$ and $r > 0$

1) $B(x, r) = \{y \in X \mid d(x, y) < r\}$

(the open ball of center x and radius r)

2) $B[x, r] = \overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$

(the closed ball of center x and radius r)

Theorem. Let (X, d) be a metric space and

$\tau_d = \{\emptyset\} \cup \{A \subset X \mid \forall\, x \in A, \exists\, r > 0 \text{ s.t. } B(x, r) \subset A\}$

The pair (X, τ_d) is a topological space

Proof:

1) $\phi \in \tau_d$ (obvious)

We want to prove $X \in \tau_d$

Let $x \in X$. For any $r > 0$ we have that $B(x, r) \subset X$.

Hence $X \in \tau_d$

2) Let $D_1, D_2 \in \tau_d$. We will show that $D_1 \cap D_2 \in \tau_d$

If $D_1 \cap D_2 = \phi$, then $D_1 \cap D_2 \in \tau_d$

Let us suppose that $D_1 \cap D_2 \neq \phi$.

Let $x \in D_1 \cap D_2$, then $x \in D_1$ and $x \in D_2$

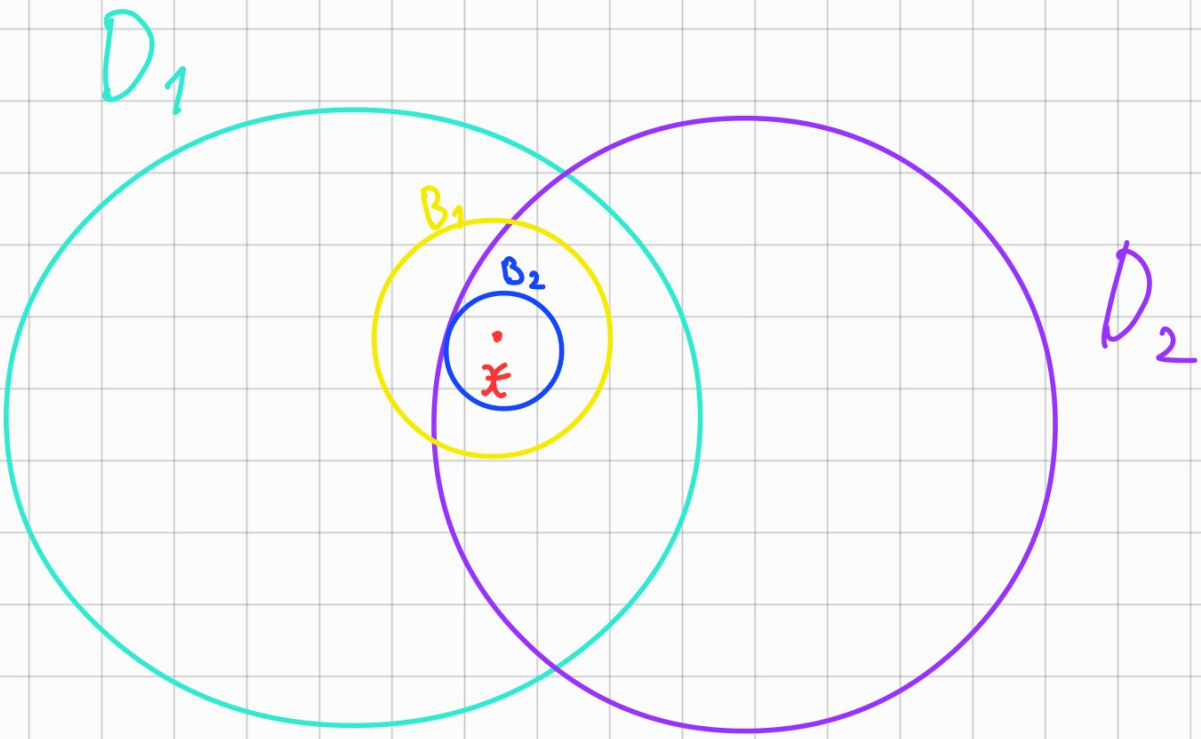
$$\left.\begin{array}{l} x \in D_1 \\ D_1 \in \tau_d \end{array}\right\} \Rightarrow \exists\, r_1 > 0 \text{ s.t. } B(x, r_1) \subset D_1$$

$$\left.\begin{array}{l} x \in D_2 \\ D_2 \in \mathcal{T}_d \end{array}\right\} \Rightarrow \exists\, r_2 > 0 \text{ s.t. } B(x, r_2) \subset D_2$$

Take $r = \min\{r_1, r_2\}$. We have that

$$B(x, r) \subset B(x, r_1) \cap B(x, r_2) \subset D_1 \cap D_2$$

For $\mathbb{R}^2$



3) Let $(D_i)_{i\in I} \subset \tau_d$. We will prove that $\bigcup_{i\in I} D_i \in \tau_d$

If $\bigcup_{i\in I} D_i = \phi$, it is clear that $\bigcup_{i\in I} D_i \in \tau_d$

Let us assume that $\bigcup_{i\in I} D_i \neq \phi$

Hence $\exists\, i_0 \in I$ s.t. $x \in D_{i_0}$

$$\left.\begin{array}{l} x \in D_{i_0} \\ D_{i_0} \in \tau_d \end{array}\right\} \Rightarrow \exists\, r_{i_0} > 0 \text{ s.t. } B(x, r_{i_0}) \subset D_{i_0}$$

We have $B(x, r_{i_0}) \subset D_{i_0} \subset \bigcup_{i\in I} D_{i_0}$

Hence $\bigcup_{i\in I} D_i \in \tau_d$

Therefore (X, τ_d) is a topological space

Def The topology $\mathcal{T}_d$ from the preceding definition is called the topology induced by the metric d.

Obs: Given a metric space (X, d), we can construct the topological space $(X, \mathcal{T}_d)$. Hence it makes sense to define open sets, closed sets, compact sets, neighborhoods, etc. in a metric space (referring to the topology induced by the metric).

Def Let (X, d) be a metric space, $(x_n)_n \subset X$ and $l \in X$. We say that l is a limit of $(x_n)_n$ with respect to the metric d and write if

$$\lim_{n\to\infty} x_n \overset{d}{=} \ell$$

or

$$x_n \xrightarrow[n\to\infty]{d} \ell$$

if $\lim_{n\to\infty} x_n \overset{\tau_d}{=} \ell$

Proposition Let (X, d) be a metric space. Then (X, τ_d) is a separated (or Hausdorff) topological space

Obs: In any metric space, every sequence can have at most one limit

Proposition Let (X, d) be a metric space, $(x_n)_n \subset X$ and $\ell \in X$. The following statements are equivalent:

1) $\lim_{n\to\infty} x_n \overset{d}{=} l$

2) $\lim_{n\to\infty} d(x_n, l) = 0$ (it is very well known that $d(x_n, l) \in [0, \infty) \; \forall n \in \mathbb{N}$)

Obs: The syntagm „with respect to the metric space d" can be replaced by the syntagm „in the metric space (X, d)".

Proposition: Let (X, d) be a metric space, $A \subset X$ and $x \in X$

1) $(x \in \mathring{A}) \Leftrightarrow (\exists r > 0$ s.t. $B(x, r) \subset A)$

2) $(x \in \overline{A}) \Leftrightarrow (\forall r > 0$, we have $B(x, r) \cap A \neq \phi)$

3) $(x \in A') \Leftrightarrow (\forall r > 0$, we have $B(x, r) \cap (A \setminus \{x\}) \neq \phi)$

4) $(x \in Fr(A)) \Leftrightarrow (x \in \overline{A} \setminus \mathring{A})$

5) $(x \in \gamma_{so}(A) = {}'A) \Leftrightarrow (x \in \overline{A} \setminus A')$

Proposition Let (X,d) be a metric space, $A \subset X$ and $x \in X$. Then:

1) $(x \in \bar{A}) \Leftrightarrow (\exists (x_n)_n \subset A$ s.t. $\lim\limits_{n\to\infty} x_n \overset{d}{=} x)$

2) $(x \in A') \Leftrightarrow (\exists (x_n)_n \subset A \setminus \{x\}$ s.t. $\lim\limits_{n\to\infty} x_n \overset{d}{=} x)$

Let us consider the metric space $(\mathbb{R}, d)$, where $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$; $d(x,y) = |x-y|$

Obs: Let $x \in \mathbb{R}$ and $r > 0$

1) $B(x,r) = \{y \in \mathbb{R} \mid d(x,y) < r\} =$

$= \{y \in \mathbb{R} \mid |x-y| < r\} =$

$= \{y \in \mathbb{R} \mid |y-x| < r\} =$

$= \{y \in \mathbb{R} \mid -r < y-x < r\} =$

$= \{y \in \mathbb{R} \mid x-r < y < x+r\}$

$= (x-r, x+r)$

2) $B[x, r] = \overline{B}(x, r) = [x - r, x + r]$

Obs: In the metric space $(\mathbb{R}, d)$:

1) The intervals of the forms:

$(-\infty, a)$; (a, ∞) and (a, b) are open sets, where $a, b \in \mathbb{R}$ and $a \leq b$

2) The intervals of the forms

$(-\infty, a]$; $[a, \infty)$ and $[a, b]$ are closed sets, where $a, b \in \mathbb{R}$ and $a \leq b$

Exercise: Perform the topological analysis of $A \subset \mathbb{R}$ (i.e. find $\mathring{A}$, $\bar{A}$, A', $\mathrm{Fr}(A)$, $\mathrm{Iso}(A)$), where:

a) $A = \mathbb{Q}$

Sol a)

1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists r > 0$ s.t. $B(x, r) \subset A$

$B(x, r) = (x - r, x + r)$, $A = \mathbb{Q}$

$\mathring{A} = \emptyset$, because between any two real numbers, we have infinitely many rational numbers and infinitely many irrational numbers.

2) $\overline{A} = ?$

$x \in \overline{A} \Leftrightarrow \forall r$, we have $B(x, r) \cap A \neq \emptyset$

$B(x, r) = (x - r, x + r)$, $A = \mathbb{Q}$

$\overline{A} \subset \mathbb{R}$ (obvious). Let $x \in \mathbb{R}$ and $r > 0$.

We have that $(x - r, x + r) \cap \mathbb{Q} \neq \emptyset$, because between any two real numbers, we have infinitely many rational numbers and

infinitely many irrational numbers.

Hence $\mathbb{R} \subset \overline{A}$

3) $A' = ?$

$x \in A' \Leftrightarrow \forall r > 0$, we have

$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$

$\quad\;\; \| \qquad\quad \|$

$(x-r, x+r) \quad \mathbb{Q} \setminus \{x\}$

$A' \subset \overline{A} = \mathbb{R}$. Let $x \in \mathbb{R}$ and $r > 0$.

We have that $(x-r, x+r) \cap (\mathbb{Q} \setminus \{x\}) \neq \emptyset$, because between any two real numbers, we have infinitely many rational numbers and infinitely many irrational numbers.

Hence $\mathbb{R} \subset A'$. So $A' = \mathbb{R}$

4) $E_3(A) = ?$

$E_3(A) = \partial A = \bar{A} \setminus \mathring{A} = \mathbb{R} \setminus \phi = \mathbb{R}$

5) $\gamma_{30}(A) = ?$

$\gamma_{30}(A) = {}^{1}A = \bar{A} \setminus A' = \mathbb{R} \setminus \mathbb{R} = \phi$ ▨

b) $A = [0, 2) \cup \{3, 4\}$

Lol b)