Notions of topology

Def. Let $X \neq \emptyset$. A set $\tau \subset \mathcal{P}(X)$

$\mathcal{P}(X) \overset{\text{def}}{=} \{A \mid A \subset X\}$

τ is called the topology on X if:

a) $\emptyset, X \in \tau$

b) $\forall D_1, D_2 \in \tau$, we have $D_1 \cap D_2 \in \tau$

c) $\forall (D_i)_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} D_i \in \tau$

Def: Let $X \neq \emptyset$ and $\tau \subset \mathcal{P}(X)$ be a topology on X. The pair (X, τ) is called a topological space.

Example:

1) Let $X \neq \phi$ and $\tau = \mathcal{P}(X)$. The pair (X, τ) is a topological space

2) Let $X \neq \phi$ and $\tau = \{\phi, X\}$. The pair is a topological space.

3) Let $X = \mathbb{R}$ and $\tau = \{\phi, \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}$

The pair (X, τ) is a topological space.

Justification

a) $\phi, \mathbb{R} \in \tau$ (obious)

b) Let $D_1, D_2 \in \tau$. We show that $D_1 \cap D_2 \in \tau$

- If $D_1 = \phi$ or $D_2 = \phi$, then $D_1 \cap D_2 = \phi \in \tau$
- If $D_1 = \mathbb{R}$, then $D_1 \cap D_2 = D_2 \in \tau$

  $D_2 = \mathbb{R}$, then $D_1 \cap D_2 = D_1 \in \tau$

- Let us assume that

$$D_1 = (-\infty, a_1)$$
$$D_2 = (-\infty, a_2) \quad ; \; a_1, a_2 \in \mathbb{R}$$

Hence $D_1 \cap D_2 = (-\infty, \min(a_1, a_2))$ and $\min(a_1, a_2) \in \mathbb{R}$ $\Rightarrow D_1 \cap D_2 \in \mathcal{T}$

c) Let $(D_i)_{i \in I} \subset \mathcal{T}$. We show that $\bigcup_{i \in I} D_i \in \mathcal{T}$

- If $\exists\, i_0 \in I$ s.t $D_{i_0} = \mathbb{R}$, then $\bigcup_{i \in I} D_i = \mathbb{R} \in \mathcal{T}$

- If $D_i = \phi \quad \forall i \in I$, then $\bigcup_{i \in I} D_i = \phi \in \mathcal{T}$

- Let us assume that $D_i = (-\infty, a_i) \; \forall i \in I$, where $a_i \in \mathbb{R} \; \forall \; i \in I$

We have $\bigcup_{i \in I} D_i = (-\infty, \sup_{i \in I} a_i) \in \mathcal{T}$

Therefore $(X, \mathcal{T})$ is a topological space.

Def: Let $(X, \mathcal{T})$ be a topological space.

1) A set $D \subset X$ is called an **open set** if $D \in \mathcal{T}$

2) A set $F \subset X$ is called a **closed set** if $X \setminus F = \underset{\text{not.}}{\complement F} \in \mathcal{T}$

the complement

3) A set $K \subset X$ is called a **compact set** if $\forall (D_i)_{i \in I} \subset \mathcal{T}$ s.t. $K \cup \bigcup_{i \in I} D_i$, $\exists\ J \subset I$, J finite s.t. $K \subset \bigcup_{i \in J} D_i$

4) Let $x \in X$. A set $V \subset X$ is called a **neighborhood** of x if $\exists\ D \in \mathcal{T}$ s.t. $x \in D \subset V$

Notation Let $(X, \mathcal{T})$ be a topological space and $x \in X$. We denote

$\mathcal{V}_x = \{V \subset X \mid V \text{ is a neighborhood of } x\}$

Topological analysis of a set

Def: Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $x \in X$. We say that x is a/an

1) Interior point of A if $A \in \mathcal{V}_x$
2) Adherent (or tolerance) point of A if, $\forall V \in \mathcal{V}_x$, we have $V \cap A = \phi$

3) Accumulation (or limit or cluster) point of A if, $\forall \; V \in \mathcal{V}_x$, we have $V \cap (A \setminus \{x\}) \neq \phi$

4) Boundary point of A if x is an adherent point of A and x is not an interior point of A

5) Isolated point of A if x is an adherent point of A and x is not an accumulation point of A

Notations In the context of the preceding definition, we denote:

1) $\mathring{A} = \{x \in X \mid x$ is an interior point of $A\}$
(the interior of A)

2) $\bar{A} = \{x \in X \mid x$ is an adherent point of $A\}$
(the adherence or the closure of A)

3) $A' = \{x \in X \mid x$ is an accumulation point of $A\}$
(the derived set of A)

4) $Fr(A) = \partial A = \{x \in X \mid x$ is a boundary point of $A\}$
(the boundary of A = "Frontiera")

5) $Iso(A) = {}'A = \{x \in X \mid x$ is an isolated point of $A\}$
(the set of all isolated points of A)

Def: Let (X, τ) be a topological space, $(x_n)_n \subset X$ and $l \in X$. We say that l is a limit of the sequence $(x_n)_n$ with respect to the topology τ and write $\lim\limits_{n\to\infty} x_n = l$ or $x_n \xrightarrow[n\to\infty]{\tau} l$ if

$\forall V \in \mathcal{V}_l$, $\exists\, n_V \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_V$, we have $x_n \in V$

Obs: The syntagm „with respect to the topology τ" can be replaced by the syntagm „in the topological space (X, τ)".

Ex Let $X = \phi$ and $\tau = \{\phi, X\}$.
For any $x \in X$, $\mathcal{V}_x = \{X\}$

Let $(x_n)_n \subset X$ and $l_1, l_2 \in X$ $(l_1 \neq l_2)$

According to the definition, we have

$\lim_{n\to\infty} x_n \overset{\tau}{=} l_1$ and $\lim_{n\to\infty} x_n \overset{\tau}{=} l_2$

Obs: In an arbitrary topological space, a sequence can have multiple limits

Def: Let (X, τ) be a topological space.
We say that (X, τ) is separated (or Hausdorff) if, $\forall\, x, y \in X$, $\exists\, U \in \mathcal{V}_x$ s.t. $V \in \mathcal{V}_y$

$U \cap V = \phi$

Proposition In any topological space (X, τ) every sequence $(x_n)_n$ has at most one limit.

Properties of the interior of a set

Let (X, τ) be a topological space, $A \subset X$, $B \subset X$

1) $\mathring{A} \subset A$

Let $x \in \mathring{A}$. Then $A \in \mathcal{V}_x$ Hence $\exists\, D \in \tau$ s.t. $x \in D \subset A$. Therefore $x \in A$, i.e. $\mathring{A} \subset A$

2) $\mathring{A} = \bigcup_{\substack{D \in \tau \\ D \subset A}} D$

"$\subset$" Let $x \in \mathring{A}$. Then $A \in \mathcal{V}_x$. Hence $\exists\, D \in \tau$ s.t. $x \in D \subset A$, i.e. $x \in \bigcup_{\substack{D \in \tau \\ D \subset A}} D$, i.e. $\mathring{A} \subset \bigcup_{\substack{D \in \tau \\ D \subset A}} D$

"⊃"

Let $x \in D$. Then $\exists D \in \mathcal{T}, D \subset A$ s.t. $x \in D \subset A$,
$D \in \mathcal{T}$
$D \subset A$

i.e. $A \in \mathcal{V}_x$, i.e. $x \in \mathring{A}$, i.e. $\bigcup_{\substack{D \in \mathcal{T} \\ D \subset A}} D \subset \mathring{A}$

Hence $\mathring{A} = \bigcup_{\substack{D \in \mathcal{T} \\ D \subset A}} D$

3) $\mathring{A}$ is an open set

By Property 2 $\mathring{A}$ is an arbitrary union belonging to $\mathcal{T}$. Hence $\mathring{A} \in \mathcal{T}$, i.e. $\mathring{A}$ is an open set.

Obs: We deduced that $\mathring{A}$ is the greatest set (in the sense of inclusion) which is open and is contained in A

4) A is open $\Leftrightarrow$ $A = \mathring{A}$

"$\Leftarrow$"

$\mathring{A}$ is open
$A = \mathring{A}$
$\Rightarrow A$ is open

"$\Rightarrow$"

A is open
$\mathring{A} \subset A$
$\mathring{A}$ is the greatest open set which is contained in A
$\Rightarrow A \subset \mathring{A}$

By property 1 we know that $\mathring{A} \subset A$

$\Rightarrow$
$\Rightarrow A = \mathring{A}$

5) If $A \subset B$, then $\mathring{A} \subset \mathring{B}$

Exercise!

6) i) $\mathring{A} \cap \mathring{B} = \overset{\circ}{\overline{A \cap B}}$

ii) $\mathring{A} \cup \mathring{B} = \overset{\circ}{\overline{A \cup B}}$

Exercise!

Obs: The inclusion in 6) ii) can be strictly

Properties of the adherence of a set

Let $(X, \mathcal{T})$ be a topological space,

$A \subset X$
$B \subset X$

1) $A \subset \bar{A}$

Let $x \in A$. We have $x \in V \cap A$, i.e.
$V \in \mathcal{V}_x$

$V \cap A \neq \phi$, i.e. $x \in \bar{A}$, i.e. $A \subset \bar{A}$

2) $\bar{A} = \bigcap_{\substack{F \text{ closed} \\ F \supset A}} \overline{F}$

„⊂" Let $x \in \bar{A}$. We show that $x \in \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$. Let us assume, by absurdity, that $\exists F \subset X$ s.t. F closed, $F \supset A$ and $x \notin F$

$$\left.\begin{array}{l} x \notin F \Rightarrow x \in CF \\ F \text{ closed} \Rightarrow CF \in \mathcal{T} \end{array}\right\} \Rightarrow CF \in \mathcal{V}_x$$

$$\left.\begin{array}{l} x \in \bar{A} \\ CF \in \mathcal{V}_x \end{array}\right\} \Rightarrow CF \cap A \neq \phi, \text{ a contradiction with } F \supset A$$

Hence $x \in \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$, i.e. $\bar{A} \subset \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$

"$\supset$" Let $x \in \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$. We show that $x \in \overline{A}$.

We assume, by absurdity, that $x \notin \overline{A}$.

Then $\exists\ V_0 \in \mathcal{V}_x$ s.t.

$V_0 \cap A = \phi$

$V_0 \in \mathcal{V}_x \Rightarrow \exists\ D_0 \in \tau$ s.t. $x \in D_0 \subset V_0$

We have $D_0 \cap A = \phi$, i.e. $A \subset C D_0$

$D_0 \in \tau \Rightarrow C D_0$ closed

But $x \in \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$, hence $x \in C D_0$,

a contradiction with $x \in D_0$

Therefore $x \in \overline{A}$, i.e. $\overline{A} \supset \bigcap_{\substack{F \text{ closed} \\ F \supset A}} F$

3) $\bar{A}$ is closed

$$C\bar{A} = C\left(\bigcap_{\substack{F \text{ closed} \\ F \supset A}} F\right) = \bigcup_{\substack{F \text{ closed} \\ F \supset A}} CF \in \mathcal{O} \Rightarrow \bar{A} \text{ closed}$$

Obs: $\bar{A}$ is the smallest set (in the sense of inclusion) which is closed and which contains A.

4) A is closed $\Leftrightarrow A = \bar{A}$

"$\Leftarrow$" $\bar{A}$ is closed, $A = \bar{A}$ $\Rightarrow A$ is closed

„⇒" A is closed

$\bar{A}$ is the smallest closed set which contains A } ⇒

$A \supset A$

$\Rightarrow A \supset \bar{A}$

By Property 1 we have that $A \subset \bar{A}$

Hence $A = \bar{A}$

5) If $A \subset B$, then $\bar{A} \subset \bar{B}$

Just. Exercise!

6) i) $\bar{A} \cup \bar{B} = \overline{A \cup B}$

ii) $\bar{A} \cap \bar{B} = \overline{A \cap B}$

Just. Exercise!

Obs The inclusion in b) ii) can be strict

Prop: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then:

1) $C\bar{A} = \overset{\circ}{C}A$

2) $C\mathring{A} = \overline{CA}$