7) $145u + 56v = 1$

$145 = 56 \cdot 2 + 33$

$56 = 33 \cdot 1 + 23$

$33 = 23 \cdot 1 + 10$

$23 = 10 \cdot 2 + 3$

$10 = 3 \cdot 3 + \boxed{1}$

$3 = 3 \cdot 1 + 0$

$\gcd(145, 56) = 1$

$1 = 10 - 3 \cdot 3$

$= 10 - 3 \cdot (23 - 2 \cdot 10)$

$= 7 \cdot 10 - 3 \cdot 23$

$= 7 \cdot (33 - 23) - 3 \cdot 23$

$= 7 \cdot 33 - 10 \cdot 23$

$= 7 \cdot 33 - 10 \cdot (56 - 33)$

$= 17 \cdot 33 - 10 \cdot 56$

$= 17(145 - 2 \cdot 56) - 10 \cdot 56$

$= 17 \cdot 145 - 44 \cdot 56$

$u_0 = 17 \quad ; \quad v_0 = -44$

$145\,u + 56\,v = 1$

$145 \cdot 17 - 56 \cdot 44 = 1$ (-)

$145(u - 17) + 56(v + 44) = 1$

$145(u - 17) = -56(v + 44)$

$56 \mid 145(u - 17)$

$56 \mid (u - 17) \Rightarrow u = 56k + 17$

Analog, $v = -145k - 44$

$\{(u,v) \in \mathbb{Z} \times \mathbb{Z} \mid 145u + 56v = 17\} =$

$= \{(56k + 17, -145k - 44) \mid k \in \mathbb{Z}\}$

2) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 - 4x + 3, & x > 1 \\ -x + 5, & x \leq 1 \end{cases}$$

$$f(x) = \begin{cases} (x-2)^2 - 1, & x > 1 \\ -x + 5, & x \leq 1 \end{cases}$$

a) f inj.? ; f surj.?

Rom.: f inj. $\Leftrightarrow \forall x_1 = x_2$, $x_1, x_2 \in \mathbb{R} \Rightarrow f(x_1) \neq f(x_2)$

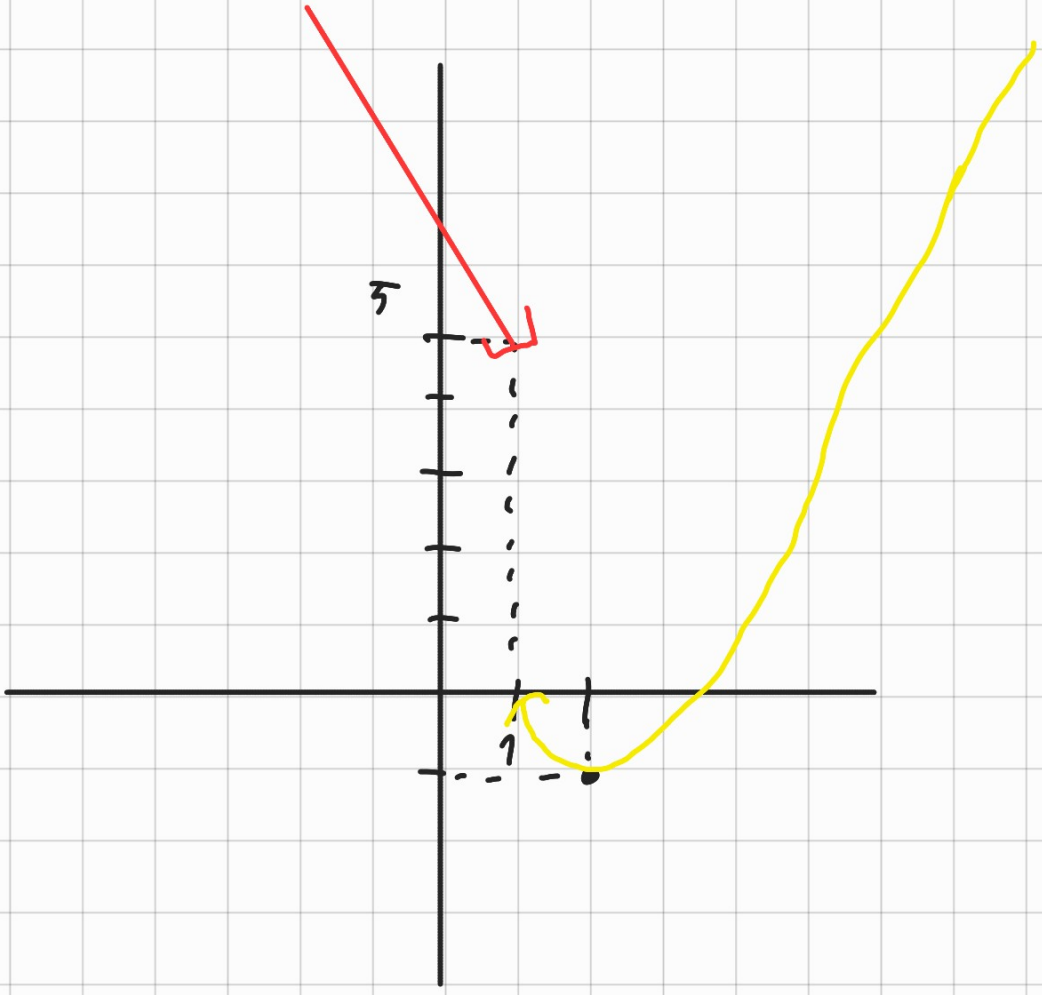
$x^2 - 4x + 3 = 0$

$\Delta = 16 - 4 \cdot 3 = 4 \Rightarrow V\left(-\frac{b}{2a}; \frac{-\Delta}{4a}\right) \to V(2, -1)$

$$x_{1,2} = \frac{4 \pm 2}{2} = \begin{cases} 1 \\ 3 \end{cases}$$

$f(1) = -1 + 5 = 4$

$f(0) = 0 + 5 = 5$



Din graf $\Rightarrow$ nu e inj.

nici surj ($\mathrm{Im} f \neq \mathbb{R}$)

b) $f((-1,4)) = ?$

$f(\overrightarrow{(-1;5)}) = ?$

$f(-1) = -(-1)+5 = 6$

$f(1) = -1+5 = 4$

$\Big\} \Rightarrow [4,6]$

$f(4) = 4^2 - 4\cdot 4 + 3 = 3 \Rightarrow (-1;3]$

$\Big\} \Rightarrow$

$\Rightarrow f((-1,4)) = (-1,3] \cup [4,6]$

$x^2 - 4x + 3 = 5 \Leftrightarrow x^2 - 4x - 2 = 0$

$\Delta = 16 + 8 = 24$

$x_{1,2} = \frac{4 \pm 2\sqrt{6}}{2}$ $\begin{cases} x_1 = 2 + \sqrt{6} \\ x_2 = 2 - \sqrt{6} \end{cases}$

$-x + 5 = -1 \Leftrightarrow x = 6 \notin -1$

$-x + 5 = 5 \Leftrightarrow x = 0 \Rightarrow (0,1]$

$$f^{-1}((-1,5)) = (0;2) \cup (2;2+\sqrt{6}) =$$

$$= (0;2+\sqrt{6}) \setminus \{2\}$$

3) Pe $\mathbb{R}^* \times \mathbb{R}^*$; „$\sim$" $(x,y) \sim (a,b) \Leftrightarrow$

$\Leftrightarrow x \cdot a > 0$ și $y \cdot b > 0$

a) „$\sim$" Rel. de echiv

R: $\forall (x,y) \in \mathbb{R}^* \times \mathbb{R}^*$

$(x,y) \sim (x,y) \Leftrightarrow x \cdot x > 0$ și $y \cdot y > 0$

$\Leftrightarrow x^2 > 0$ și $y^2 > 0$ Adev pt. $\forall (x,y) \in (\mathbb{R}^*)^2$

S: $\forall (x,y), (a,b) \in \mathbb{R}^* \times \mathbb{R}^*$ cu $(x,y) \sim (a,b) \Leftrightarrow$

$\Rightarrow xa > 0$ și $yb > 0 \Leftrightarrow ax > 0$ și $by > 0 \Leftrightarrow$

$\Leftrightarrow (a,b) \sim (x,y)$

T: $\forall\ (x,y),(a,b),(m,n) \in \mathbb{R}^* \times \mathbb{R}^*$ cu

$(x,y) \sim (a,b) \Rightarrow xa > 0\ ;\ yb > 0$

$(a,b) \sim (m,n) \Rightarrow am > 0\ ;\ bn > 0$

$xa^2m > 0\ ;\ yb^2n > 0$

$a^2 > 0\ ;\ b^2 > 0 \qquad a,b \in \mathbb{R}^*$ $\Big\}\in!$

$\Leftrightarrow xm > 0\ ;\ yb > 0\ \Leftrightarrow!$

$\Leftrightarrow (x,y) \sim (m,n)$

$R, S, T \Rightarrow$ „$\sim$" este rel. de simetrie

b) $[(-1;5)]$

Fie $(a,b) \in \mathbb{R}^* \times \mathbb{R}^*$

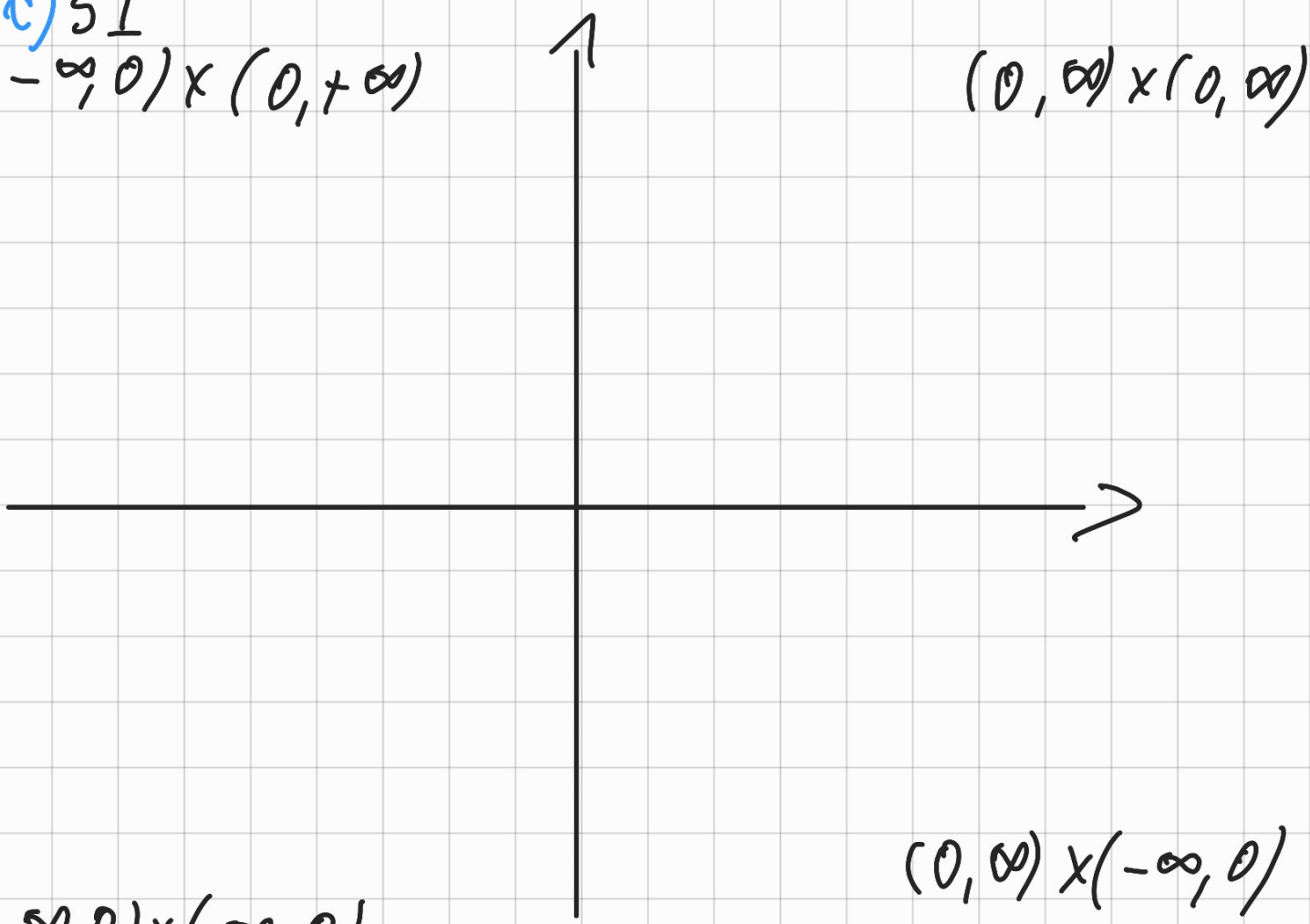
$(-1;5) \sim (a,b) \Leftrightarrow -1\,a > 0 ; 5\,b > 0 \Rightarrow$

$\Rightarrow a < 0$ și $b > 0 \Rightarrow$

$\Rightarrow [(-1;5)] = \{(a,b) \mid a \in (-\infty;0);\ b \in (0,+\infty)\}$

c) SI

$(-\infty;0) \times (0,+\infty)$



$(0,\infty) \times (0,\infty)$

$(0,\infty) \times (-\infty,0)$

$(-\infty,0) \times (-\infty,0)$

$S = \{(1,1);(-1,1);(1,-1);(-1,-1)\}$

Prob: Pe $\mathbb{R}$, $x \sim y \Leftrightarrow (x^2+2x+5)^2 = (y^2+2y+5)^2$

$\Leftrightarrow f(x) = f(y)$

Fie $f: \mathbb{R} \to \mathbb{R}$; $f(x) = (x^2+2x+5)^2$

a) $R: x \sim x$, $f(x) = f(x)$

$S: x \sim y \Leftrightarrow f(x) = f(y) \Leftrightarrow f(y) = f(x) \Leftrightarrow y \sim x$

$T: x \sim y \Leftrightarrow f(x) = f(y)$
$y \sim z \Leftrightarrow f(y) = f(z)$
$\Leftrightarrow f(x) = f(z) \Leftrightarrow x \sim z$

c) S ; $S = Im(f) = [16, +\infty)$ (prop. de universalitate)

$\min(x^2+2x+5) = \frac{-\Delta}{4a}$; $f(-\frac{b}{2a})$

$(x+1)^2 + 4 \geq 4$

$(x+1)^2 \geq 0$

d) $\mathbb{R}/_{\sim} \simeq [16, +\infty)$ (Mereu cf. prop. de universalitate)

↑ să fie bijecție

$F: \mathbb{R}/_{\sim} \to [16, +\infty)$; $F(\hat{x}) = f(x)$ bij + bine def

F - bine def. $\Leftrightarrow F(\hat{x}) = F(\hat{y})$ când $x \sim y$

$\mathbb{R}/_{\sim}$ = mulțimea factor, mulțimea claselor de echivalență

Ex seminar

Aflați $17^{200} \bmod 29$

1) $a^{p-1} \equiv 1 \pmod p$; $\gcd(a,p)=1$

Mica TR. a lui Fermat

2) $a^{\varphi(n)} \equiv 1 \pmod n$, $\gcd(a,n)=1$

$\varphi(n)$ – câte nr. $< n$ sunt coprime cu n

6 : 1, 5
$\varphi(6)=2$

TR. lui Euler

$\varphi(p) = p-1$

↑ prim

$\varphi(n) =$

$n = p_1^{\ell_1} \cdot p_2^{\ell_2} \cdot \ldots \cdot p_k^{\ell_k}$,

$\varphi(n) = n\left(1-\frac{1}{\ell_1}\right)\left(1-\frac{1}{\ell_2}\right)\cdot\ldots\cdot\left(1-\frac{1}{\ell_k}\right)$

I) Exponențiere rapidă:

$$17^{200} = 17^{100} \cdot 17^{100}$$

$$17^{100} = 17^{50} \cdot 17^{50}$$

$$17^{50} = 17^{25} \cdot 17^{25}$$

$$17^{25} = 17 \cdot 17^{24}$$

$$17^{24} = 17^{12} \cdot 17^{12}$$

$$17^{12} = 17^{6} \cdot 17^{6}$$

$$17^{6} = 17^{3} \cdot 17^{3}$$

II) Aplicați Fermat / Euler

Fermat: $a^{p-1} \equiv 1 \mod p$

$a \leftarrow 17$, $p \leftarrow 29$

$$17^{28} \equiv 1 \mod 29$$

$$200 = 7 \cdot 28 + 4$$

$$17^{200} = 17^{7 \cdot 28 + 4} \equiv 17^{7 \cdot 28} \cdot 17^{4} \pmod{29}$$

$$= (17^{28})^7 \cdot 17^4$$

$$= 1 \cdot 17^4$$

$$= 17^4$$

Ex original: Aflați restul împărțirii lui

$N = 774^{248}$ la 7

$774^{148} \mod 7$

Ex modificat: Aflați restul împărțirii lui

$N = 774^{248}$ la 35

Nu merge $a^{p-1} \equiv 1 \mod p$ fiindcă

35 NU e prim

Euler: $a^{\varphi(n)} \equiv 1 \pmod{n}$; $\gcd(a,n)=1$

$\varphi(n) = \varphi(35) = 35\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)$

$35 = 5^1 \cdot 7^1$ $= 35 \cdot \frac{4}{5} \cdot \frac{6}{7}$

$= 24$

$774^{24} \equiv 1 \pmod{35}$

$148 = k \cdot 24 + 4$

$= 6 \cdot 24 + 4$

$(774)^{6\cdot 24+4} = 774^{6\cdot 24} \cdot 774^4$

$= (774^{24})^6 \cdot 774^4$

$= 1^6 \cdot 774^4$

$= 774^4$

$774 \underset{4}{\equiv} 4$

$= 4^4 = 256 \bmod 35$

Temă: $1^{321}23 \mod 7$

Ex test seria 13

$f: A \to B$; $A_1, A_2 \subseteq A$

Arr. că $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

Rez: Dublă Incluziune:

A, B - mulțimi

$A \subseteq B$ și $B \subseteq A \Rightarrow A = B$

I $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$

Mod general: $A \subseteq B$

Alg un x oarecare din A și arăt că $x \in B$

Fie $y \in f(A_1 \cup A_2)$

$y = f(x) \Rightarrow x \in A_1 \cup A_2 \Rightarrow$

$\Rightarrow x \in A_1$ sau $x \in A_2 \Rightarrow$

$\Rightarrow f(x) \in f(A_1)$ sau $f(x) \in f(A_2) \Rightarrow$

$\Rightarrow y \in f(A_1)$ sau $y \in f(A_2)$

$\Rightarrow y \in f(A_1) \cup f(A_2) \Rightarrow$

$\Rightarrow f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$ (1)

II) $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$

Fie $y \in f(A_1) \cup f(A_2) \Rightarrow$

$y = f(x)$

$f^{-1}(y) = x$

$\Rightarrow y \in f(A_1)$ sau $y \in f(A_2) \Rightarrow$

$\Rightarrow x \in A_1$ sau $x \in A_2 \Rightarrow$

$\Rightarrow x \in A_1 \cup A_2$

$\Rightarrow f(x) \in f(A_1 \cup A_2) \Rightarrow$

$\Rightarrow f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$ (2)

(1), (2) $\Rightarrow f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$