Def: Let $M \neq \emptyset$ A relation ρ on M

1) is called an order relation on M if it is

→ reflexive: $\forall x \in M \quad x \rho x$

→ antisymmetric: $\forall x, y \in M \quad x \rho y$ and $y \rho x \Rightarrow$
$\Rightarrow x = y$

→ transitive: $\forall x, y, z \in M \quad x \rho y$ and $y \rho z \Rightarrow$
$\Rightarrow x \rho y$

We say (M, ρ) is a (partially) ordered set (poset)

2) ρ is a **total** ordering on M if

moreover: any 2 elements $x, y \in M$ are comparable, i.e. $x \rho y$ or $y \rho x$

Usually for order relations one uses „$\leq$"

Examples: i) $(\mathbb{R}, \leq)$ is a totally ordered set
↳ the usual ordering $x \leq y \iff$

$\iff \exists\, a \in \mathbb{R}_+$ s.t. $x + a = y$

ii) The inverse of an order relation is again an order relation

$$\leq^{-1} =: \geq$$

iii) $(M, \leq)$ ordered set

Define „$<$" on M

$x < y \iff x \leq y$ and $x \neq y$

$\leq \;=\; < \cup \Delta_M$

iv) M set $(\mathcal{P}(M), \subseteq)$

For $A, B \in \mathcal{P}(M)$: $A \subseteq B$ is an order rel.
Subsets of M
↳ R: $A \subseteq A$
↳ A: if $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$
↳ T: if $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

If $|M| \geq 2$ $x \in M$ $\{x\} \not\subseteq \{y\}$
$y \in M$ $\{y\} \not\subseteq \{x\}$

So $\{x\}, \{y\} \in \mathcal{P}(M)$ are not comparable so the ordering „$\subseteq$" is not <u>total</u>

$(M, \leq)$ order relation on M

For $(x, y) \in M$: we say $x \lessdot y$ if $x < y$
(x is covered by y)

and $\forall\, z \in M$ while $x \leq z \leq y \Rightarrow \begin{cases} z = x \\ \text{or} \\ z = y \end{cases}$

„$<$" $:= \leq \setminus \Delta_M$

„$\lessdot$"

Representation of ordered set
Hasse diagrams

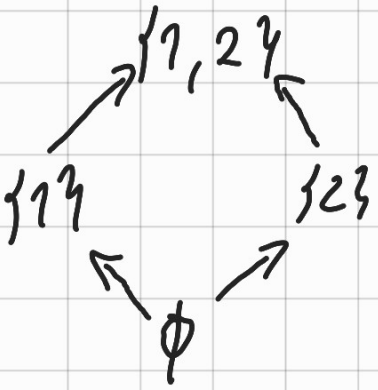
$(M, \leq)$ ordered set $\rightsquigarrow$ oriented graph

- vertices / nodes: M
- arrows $x \to y$ if $x \lessdot y$

Example:

1) $(\mathcal{P}(\{1,2\}), \subseteq)$

$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



2) When $(M, \leq)$ is a totally ordered set



The Hasse diagram is a „chain"

Def: A subset $N \subseteq (M, \leq)$ is called a **chain** if $(N, <)$ is totally ordered set

Def: Let $(M_1, \leq_1), (M_2, \leq_2)$ be two ordered sets

- The map $f: M_1 \to M_2$ is called **monotone** (isotone) if
  $\forall\, a, b \in M_1 \quad a \leq_1 b \Rightarrow f(a) \leq_2 f(b)$

- The map $f: M_1 \to M_2$ is called **isomorphism** of ordered sets if
  - $\to$ f is bijective
  - $\to$ f and f^{-1} are monotone

Remark: If f is bij. + f monotone $\not\Rightarrow$ f^{-1} is monotone

Special elements in ordered sets

Let $(M, \leq)$ ordered set

$x \in M$ is called

1) a first element (or a MINIMUM) in M if $\forall y \in M \quad x \leq y$
2) a last element (or a MAXIMUM) in M if $\forall y \in M \quad y \leq x$

3) a minimal element in M if $(\nexists y \in M$ with $y < x) \Leftrightarrow$

$\Leftrightarrow \forall y \in M \quad y \leq x \Rightarrow y = x$

4) a maximal element in M if $(\nexists y \in M$ with $x < y) \Leftrightarrow$

$\Leftrightarrow \forall y \in M \quad x \leq y \Rightarrow x = y$

$(M, \leq)$

Prop 1) If x is a MINIMUM in $(M, \leq)$, then it is unique

Proof: Assume x_1, x_2 are 2 minimal in M

$x_1 \leq x_2$ because x_1 is a minimum
$x_2 \leq x_1$ because x_2 is a minimum
$\Rightarrow$

$\Rightarrow x_1 = x_2 \Rightarrow$ uniqueness

2) If x is a MAXIMUM, then it is unique

3) If x is a MINIMUM in M, then it is (the only minimal element in M)

Proof: → we prove x is minimal:
Let $y \in (M, \subseteq)$ with $y \subseteq x$ } $\Rightarrow x = y \Rightarrow$
x minimum

$\Rightarrow x$ is minimal

→ x is the only minimal element in M

4) If x is a maximum in M, then it is only maximal element in M

$(M, \subseteq) = (\mathcal{P}\{1,2\} \setminus \{\phi\}, \subseteq)$

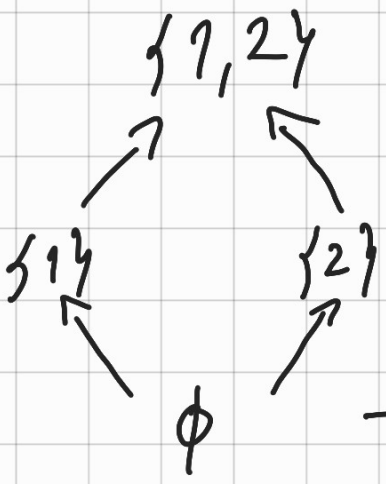
$\{\{1\}, \{2\}, \{1,2\}\}$

$\{1,2\}$ → the maximum

$\{1\}$ $\{2\}$ → two minimal elements

$\nexists$ minimum

- $(\mathcal{P}(\{1,2\}), \subseteq)$



→ the maximum

→ the minimum, also a minimal

- $(\mathbb{N}, \leq)$: 0 is the minimum

  ∄ maximal element