Def: Let $M \neq \phi$. A relation ρ on M is called an **equivalence relation** on M if it is reflexive, symmetric and transitive.

Defn If ρ is an equivalence on M, for any $x \in M$ denote $\hat{x} = \{y \in M \mid y \rho x\}$ (or $[x]$, or $\bar{x}$)

is called equivalence class of x

Denote $M/\rho = \{\hat{x} \mid x \in M\}$ is called the **quotient** space of M (w.r.t. ρ) (or the factor set)

The map $\tilde{\pi}: M \to M/\rho$

$$x \longmapsto \vec{x}$$

is surjective and is called "the canonical projection / surjection w.r.t. ρ"

A set $S \subseteq M$ is called a (full) system of representatives for ρ if S contains exactly one element from each equivalent class.

Prop: ρ is an equivalence relation on M

1) $\forall \; x \in M \qquad x \in \vec{x}$

($x \; \rho \; x$ by reflexivity of ρ)

2) Let $x, y \in M$

Then $x \not\sim y \Leftrightarrow \hat{x} = \hat{y}$

$x \sim y \Leftrightarrow \hat{x} \cap \hat{y} = \phi$

So, any 2 equivalence classes are either disjoint, or they are

Proof:

- $x \sim y \Leftrightarrow \hat{x} = \hat{y}$

"$\Rightarrow$" Assume $x \sim y$. To prove $\hat{x} = \hat{y}$"

Let $z \in \hat{x} \Rightarrow z \sim x \overset{(*)}{\Rightarrow} z \sim y \Rightarrow z \in \hat{y}$. So $\hat{x} \subseteq \hat{y}$

Let $w \in \hat{y} \Rightarrow w \sim y$ } $\Rightarrow w \sim x \Rightarrow \hat{y} \subseteq \hat{x}$
$y \sim x$

$\Rightarrow \hat{x} = \hat{y}$

"$\Leftarrow$" Assume $\vec{x} = \vec{y}$ and $x \in \vec{x}$ $\Rightarrow x \in \vec{y} \Rightarrow y \, S \, x \overset{S}{\Rightarrow} x \, S \, y$

$x \not S y \Leftrightarrow \vec{x} \cap \vec{y} = \phi$

"$\Rightarrow$" Assume $x \not S y$, i.e. $(x, y) \in S$

Assume by contradiction that $\exists z \in \vec{x} \cap \vec{y}$

$\Rightarrow z \in \vec{x}$ and $z \in \vec{y} \Rightarrow$

$\Rightarrow z \, S \, x$ and $z \, S \, y$

$\Downarrow$ ⑤

$x \, S \, z$

$\Downarrow$ ①

$x \, S \, y$ contradiction

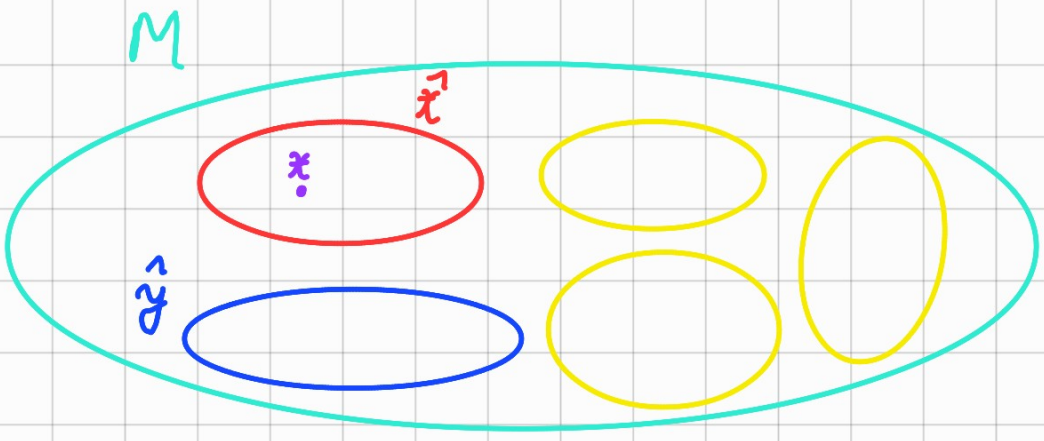
So $\vec{x} \cap \vec{y} = \phi$

"$\Leftarrow$" Assume $\vec{x} \cap \vec{y} = \emptyset$

By contradiction, assume $x \sim y \Rightarrow \vec{x} = \vec{y}$ $\Rightarrow$

$\Rightarrow \vec{x} = \emptyset$, false since $x \in \vec{x}$

3) $\bigcup_{x \in M} \vec{x} = M$



$M/\sim = \{$  $\}$

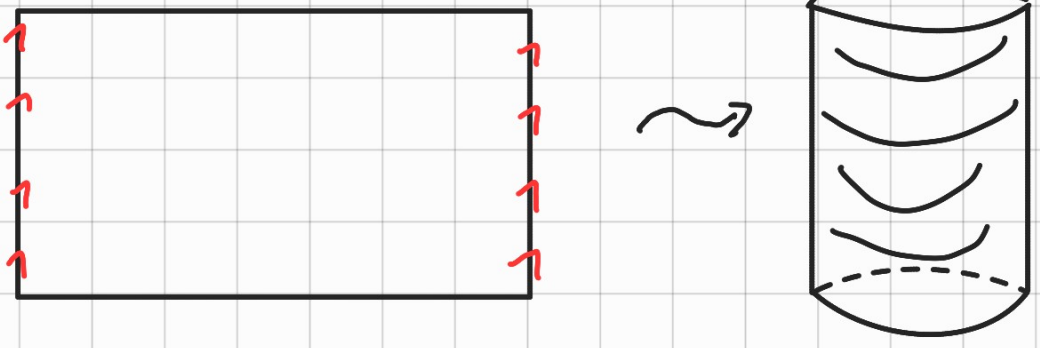
Fact: The equivalence classes $\vec{x}$ realize a partition of the set M

Defn: $M = \phi$. A partition of M is an indexed family, $(A_i)_{i \in I}$

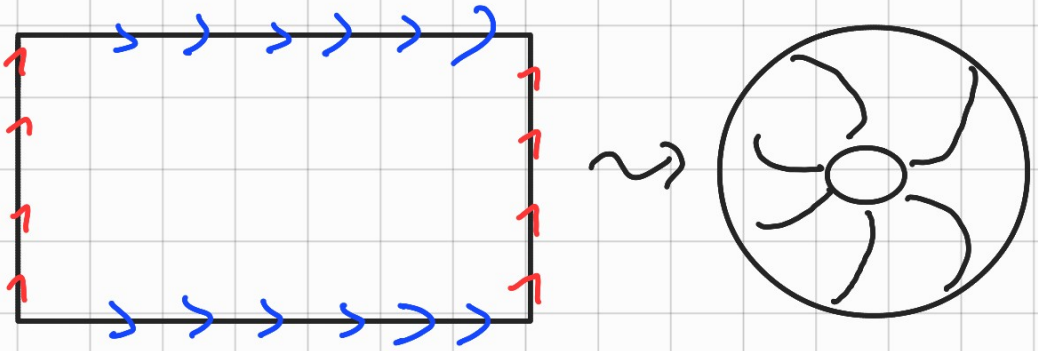
s.th.:
1) $A_i \neq \phi$, $\forall\, i \in I$
2) $A_i \cap A_j = \phi$, $\forall\, i = j$ in I
3) $\bigcup_{i \in I} A_i = M$

Example

1)

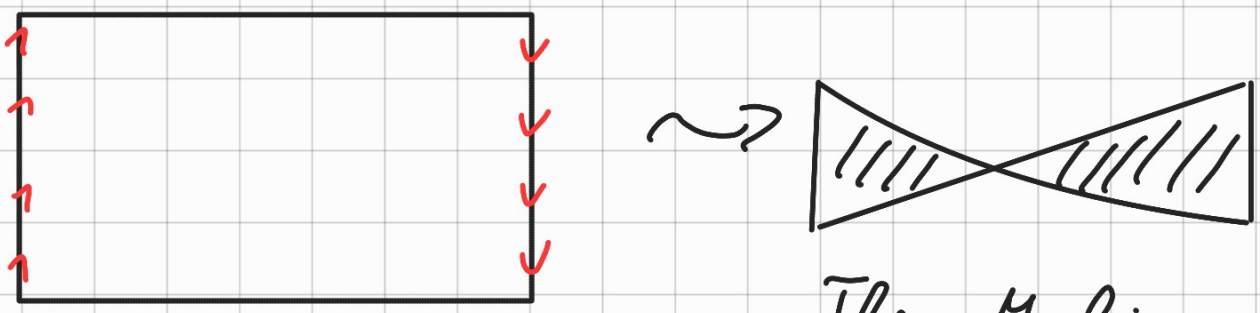


2)



quotient space

3)



The Mobious space

4 Fix $n \in \mathbb{N}^*$. On $\mathbb{Z}$ define $\sim_n$

$x, y \in \mathbb{Z}$ $\quad x \sim_n y \iff n \mid x - y$

Facts:

1) $\sim_n$ is an equivalence relation on $\mathbb{Z}$

Pf: 1) R: $\forall x \in \mathbb{Z}$ $\quad x \sim_n y$ because $n \mid x - x$
$n \mid 0$ true

2) S: Let $x, y \in \mathbb{Z}$: $x \sim_n y \Rightarrow n \mid x - y$

$\exists z \in \mathbb{Z}$ $\quad x - y = n \cdot z$

$y - x = n \cdot (-z) \Rightarrow$

$\Rightarrow n \mid y - x$

$\Rightarrow y \sim_n x$

So $\sim_n$ is symmetric relation

3) T. Let $x, y, z \in \mathbb{Z}$: $\begin{cases} x \sim_n y \\ y \sim_n z \end{cases} \Rightarrow$

$\Leftrightarrow \begin{cases} n \mid x-y \\ n \mid y-z \end{cases} \Leftarrow \begin{cases} \exists\, k_1 \in \mathbb{Z} \quad n \cdot k_1 = x-y \\ \exists\, k_2 \in \mathbb{Z} \quad n \cdot k_2 = y-z \end{cases} \Rightarrow$ (*)

$\Rightarrow n(k_1+k_2) = x-z \Rightarrow n \mid x-z$, so $x \sim_n y$, so (T)

o.k. a $x \equiv y \mod n$

$x \overset{n}{\equiv} y$

2) For $x \in \mathbb{Z}$; $\overline{x} = \{y \in \mathbb{Z} \mid y \sim_n x$
$n \mid y-x$

$\overline{x} = \{x + k \cdot n \mid k \in \mathbb{Z}\}$ $\exists k \quad y-x = k \cdot n$

3) $\overline{x} = \overline{y} \overset{\text{Prop}}{\Leftrightarrow} x \sim_n y \Leftrightarrow n \mid x-y$

System of representations:

$\hat{0} = \{0 + k \cdot n \mid k \in \mathbb{Z}\}$ → all multiples of n

$\hat{1} = \{1 + k \cdot n \mid k \in \mathbb{Z}\}$ → —— " —— $+1$

...

$\widehat{n-1}$

$\hat{n} = \{n + k \cdot n \mid k \in \mathbb{Z}\} = \hat{0}$

$\{0, 1, 2, \ldots, n-1\}$ is a system of representatives for $\sim_n$

Quotient space $\mathbb{Z}_n / \sim_n = \{\hat{x} \mid x \in \mathbb{Z}\} = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{n-1}\} = \mathbb{Z}_n$ notation

In fact $(\mathbb{Z}_n, +)$ is a ring)

$\hat{0}$ in $\mathbb{Z}_5$ is $\hat{0} = \{5k \mid k \in \mathbb{Z}\}$

$\hat{0}$ in $\mathbb{Z}_3$ is $\hat{0} = \{3k \mid k \in \mathbb{Z}\}$

Example: Let $f: A \to B$ be any map

On A we define the relation $\sim_f$ by

$$\forall\, x_1, x_2 \in A: x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$$

Then 1) $\sim_f$ is an equivalance relation on A

Reflexivity: $\forall\, x \in A \quad x \sim_f x \Leftrightarrow f(x) = f(x)$

Symmetry: Let $x_1, x_2 \in A$ s. t. $\Rightarrow$

$$\Rightarrow f(x_1) = f(x_2) \Rightarrow f(x_2) = f(x_1) \Rightarrow$$

$$\Rightarrow x_2 \sim_f x_1$$

Transitivity: Let $x_1, x_2, x_3 \in A$ s. t.

$x_1 \sim_f x_2$ and $x_2 \sim x_3 \Rightarrow$

$$\Rightarrow f(x_1) = f(x_2) = f(x_3) \Rightarrow$$

$$\Rightarrow x_1 \sim_f x_3$$

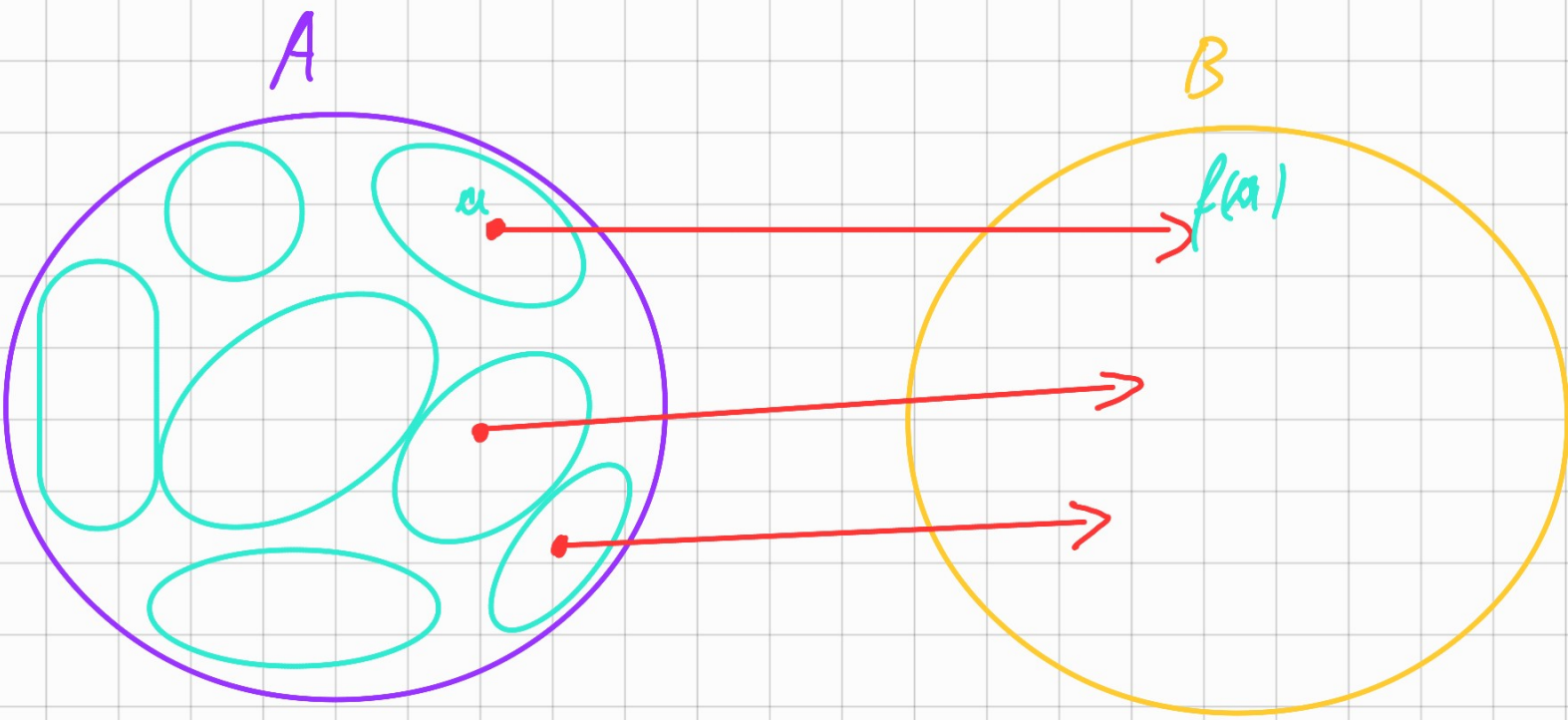
2) For $a \in A$ $\hat{a} = ?$

$\hat{a} = ?$ $\quad \hat{a} = \{x \in A \mid x \sim_f a\} = f^{-1}(f(a))$

$f(x) = f(a)$

3) There is a bijection $\varphi : A/\sim_f \to \operatorname{Im} f$ given by $\varphi(\hat{a}) = f(a)$

$A/\sim_f = \{\hat{a} \mid a \in A\}$



Example:
$\mathbb{N} \subset \mathbb{Z}$

On $\mathbb{N} \times \mathbb{N}$ define $\sim$:

$(a,b) \sim (c,d) \iff a+d = b+c$

1) [Exercise]: $\sim$ is an equivalance relation on $\mathbb{N} \times \mathbb{N}$

2) For any $(a,b) \in \mathbb{N} \times \mathbb{N}$ there exists unique $x \in \mathbb{N}$ such that $(a,b) \sim (x,0)$ or $(a,b) \sim (0,x)$

Remark: If $(a,b) \sim (x,0)$ and $(a,b) \sim (0,y)$

$\Downarrow$ $\quad$ $\Downarrow$

$a+0 = b+x \qquad a+y = b+0$

$a = (a+y)+x$

$0 = y+x \implies x = y = 0$

$x, y \in \mathbb{N}$

3) Consider the map

$$i : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \overset{\text{not}}{=} \mathbb{Z}$$
$$a \mapsto (a, 0)$$

is injective / we may view $\mathbb{N}$ as a subset of $\mathbb{N} \times \mathbb{N} / \sim$

$\widehat{(x, 0)}$ → nonnegative integers

$\widehat{(0, x)}$ → negative integers

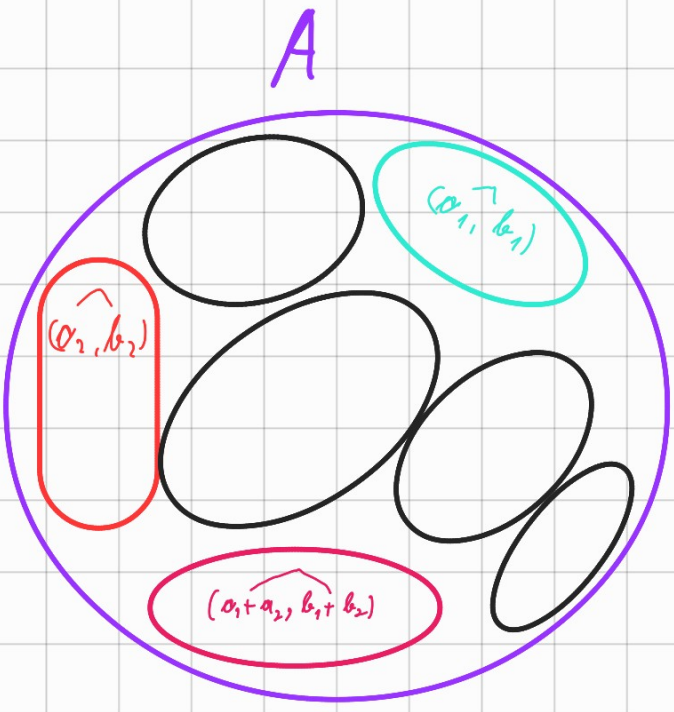
Introduce

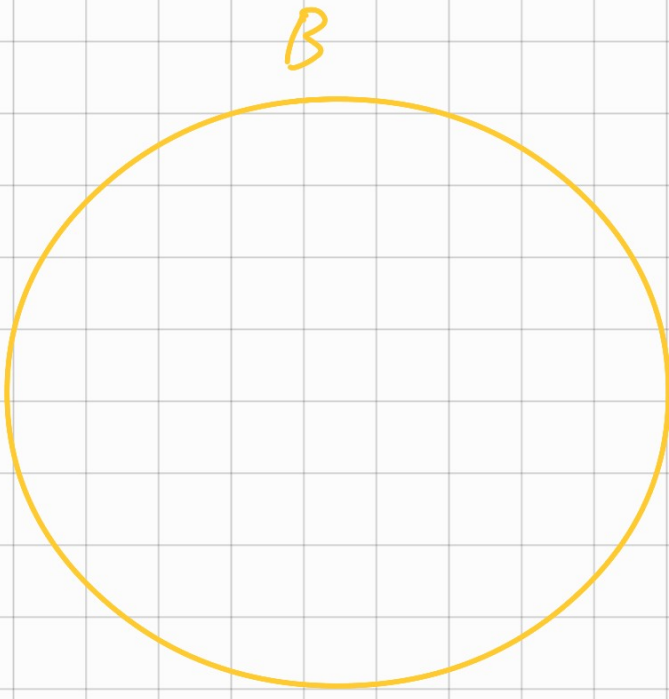
- addition ⊕ on $\mathbb{Z}$: $\widehat{(a_1, b_1)} + \widehat{(a_2, b_2)} := \widehat{(a_1 + a_2, b_1 + b_2)}$

- multiplication ⊙ on $\mathbb{Z}$: $\widehat{(a_1, b_1)} \cdot \widehat{(a_2, b_2)} = \widehat{(a_1 \cdot a_2, b_1 \cdot b_2)}$

$(\mathbb{Z}, +, \cdot)$ ring; Want That these operators on $\mathbb{Z}$ extend the ones on $\mathbb{N}$

We check $i(a+b) = i(a) + i(b)$ $\forall$ $a, b \in \mathbb{N}$
$i(a \cdot b) = i(a) \cdot i(b)$





"the $+, \cdot$: do not depend on the representatives we chose from the equivalence classes (bubbles)"

Proposition (The universal property of the quotient space)

Let $\sim$ be an equivalance relation on $M \neq \emptyset$, and $f: M \to B$ (B any set)

Then the assignement $\overline{f}: M/\sim \longrightarrow B$ defines

$$\hat{x} \longrightarrow f(x)$$

well defined map $\forall x_1, x_2 \in M$

$$x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$$

Example: $f: \mathbb{Z} \to \mathbb{Z}$

$$x \longmapsto x^3$$

$\equiv \mod 5$ an equivalance space on $\mathbb{Z}$

$$\overline{f}: \mathbb{Z}_5 = \mathbb{Z}/_{\equiv \mod 5} \longrightarrow \mathbb{Z}$$

$$\hat{x} \longmapsto \hat{x}^3$$

$0 \equiv 5 \mod 5$, but $\left.\begin{matrix} f(0) = 0^3 = 0 \\ f(5) = 5^3 = 125 \end{matrix}\right\} \not\Rightarrow$

$\Rightarrow f(0) \neq f(5)$

$\mathbb{N} \to \mathbb{Z}$

$\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^* / \sim$

$\uparrow$

"$\frac{2}{3}$" $= \frac{4}{6}$

$(2,3) \sim (4,6)$

$\frac{a}{b} = \frac{c}{d} \quad (\Rightarrow a \cdot d = b \cdot c$

$b, d \neq 0$

$a, c \in \mathbb{Z}$

On $\mathbb{Z} \times \mathbb{Z}^*$ define

$(a, b) \sim (c, d)$ if $a \cdot d = b \cdot c$

Then $\sim$ is an equivalance relation on $\mathbb{Z} \times \mathbb{Z}^*$ $\overset{\text{not}}{=} \mathbb{Q}$

$\to$ the quotient space

$\mathbb{Z} \to \mathbb{Q}$

$n \mapsto \frac{n}{1} = (n, 1)$

HWK: Define $+, \cdot$ on $\mathbb{Q}$ s.t. in the end

$(\mathbb{Q}, +, \cdot)$ a Field (Körp)