Def: Let $M \neq \phi$. A relation ρ on M is called an **equivalence relation** on M if it is reflexive, symmetric and transitive.

Defn If ρ is an equivalence on M, for any $x \in M$ denote $\hat{x} = \{y \in M \mid y \rho x\}$ (or $[x]$, or $\bar{x}$)

is called equivalence class of x

Denote $M/\rho = \{\hat{x} \mid x \in M\}$ is called the **quotient** space of M (w.r.t. ρ)

(or the factor set)

The map $\tilde{\pi}: M \to M/\rho$

$$x \longmapsto \tilde{x}$$

is surjective and is called "the canonical projection / surjection w.r.t. ρ"

A set $S \subseteq M$ is called a (full) system of representatives for ρ if S contains exactly one element from each equivalent class.

Prop: ρ is an equivalence relation on M

1) $\forall\ x \in M \qquad x \in \tilde{x}$

($x \rho x$ by reflexivity of ρ)

2) Let $x, y \in M$

Then $x \not\sim y \Leftrightarrow \hat{x} = \hat{y}$

$x \sim y \Leftrightarrow \hat{x} \cap \hat{y} = \phi$

So, any 2 equivalence classes are either disjoint, or they are

Proof:

- $x \sim y \Leftrightarrow \hat{x} = \hat{y}$

"$\Rightarrow$" Assume $x \sim y$. To prove $\hat{x} = \hat{y}$"

Let $z \in \hat{x} \Rightarrow z \sim x \overset{(*)}{\Rightarrow} z \sim y \Rightarrow z \in \hat{y}$. So $\hat{x} \subseteq \hat{y}$

Let $w \in \hat{y} \Rightarrow w \sim y$ $\Rightarrow w \sim x \Rightarrow \hat{y} \subseteq \hat{x}$
$y \sim x$

$\Rightarrow \hat{x} = \hat{y}$

"$\Leftarrow$" Assume $\vec{x} = \vec{y}$ and $x \in \vec{x}$ $\Rightarrow x \in \vec{y} \Rightarrow y \, S \, x \overset{S}{\Rightarrow} x \, S \, y$

$x \not S y \Leftrightarrow \vec{x} \cap \vec{y} = \phi$

"$\Rightarrow$" Assume $x \not S y$, i.e. $(x, y) \in S$

Assume by contradiction that $\exists z \in \vec{x} \cap \vec{y}$

$\Rightarrow z \in \vec{x}$ and $z \in \vec{y} \Rightarrow$

$\Rightarrow z \, S \, x$ and $z \, S \, y$

$\Downarrow$ ⑤

$x \, S \, z$

$\Downarrow$ ①

$x \, S \, y$ contradiction

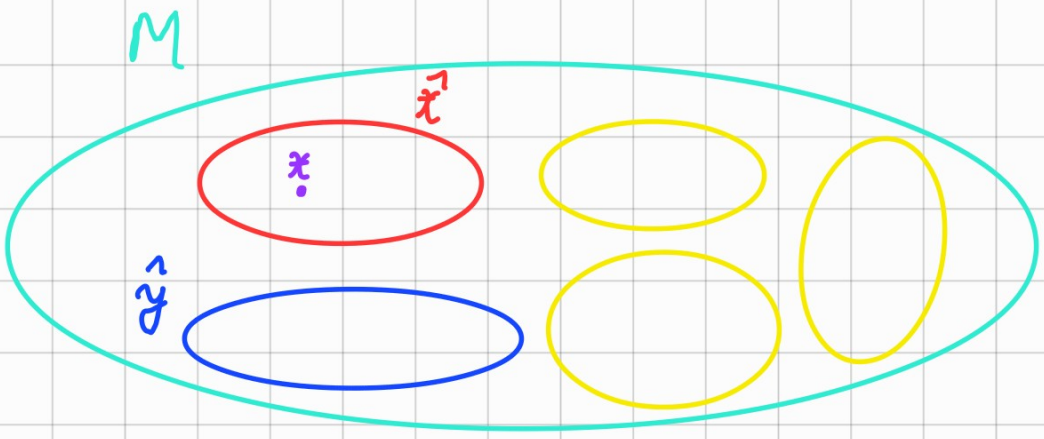
So $\vec{x} \cap \vec{y} = \phi$

"$\Leftarrow$" Assume $\vec{x} \cap \vec{y} = \emptyset$

By contradiction, assume $x \sim y \Rightarrow \vec{x} = \vec{y}$ $\Rightarrow$

$\Rightarrow \vec{x} = \emptyset$, false since $x \in \vec{x}$

3) $\bigcup_{x \in M} \vec{x} = M$



$M/\sim = \{$  $\}$

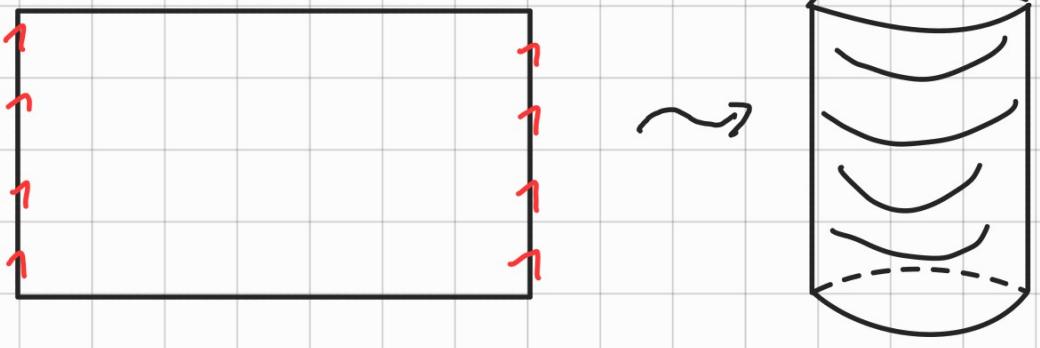
Fact: The equivalence classes $\vec{x}$ realize a partition of the set M

Defn: $M = \phi$. A partition of M is an indexed family, $(A_i)_{i \in I}$

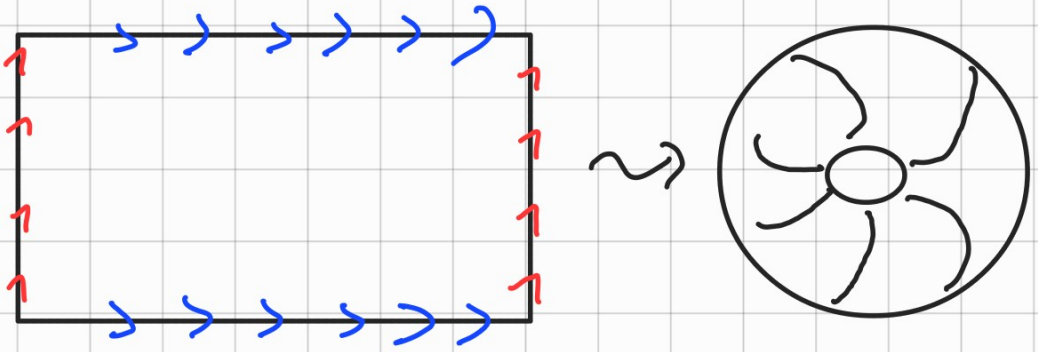
s.th.:
1) $A_i \neq \phi$, $\forall\, i \in I$
2) $A_i \cap A_j = \phi$, $\forall\, i = j$ in I
3) $\bigcup_{i \in I} A_i = M$

Example

1)

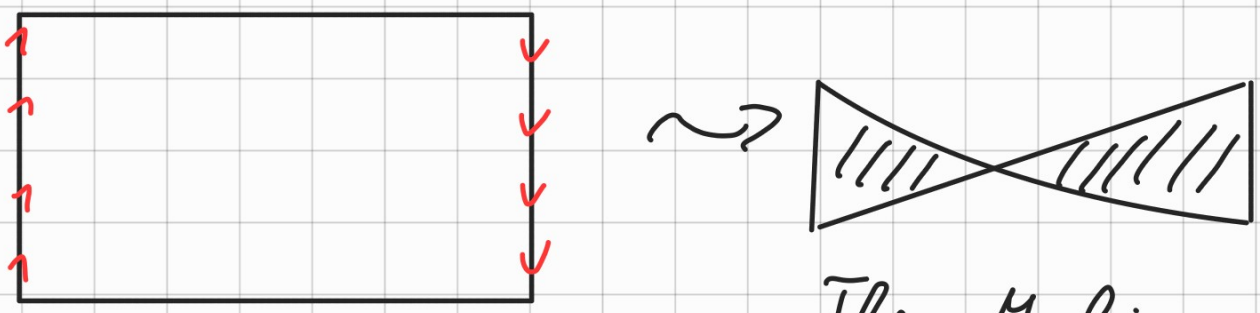


2)



quotient space

3)



The Mobious space

4 Fix $n \in \mathbb{N}^*$. On $\mathbb{Z}$ define $\sim_n$

$x, y \in \mathbb{Z}$ $\quad x \sim_n y \iff n \mid x - y$

Facts:

1) $\sim_n$ is an equivalence relation on $\mathbb{Z}$

Pf: 1] R: $\forall x \in \mathbb{Z}$ $\quad x \sim_n y$ because $n \mid x - x$
$n \mid 0$ true

2] S: Let $x, y \in \mathbb{Z}$: $x \sim_n y \Rightarrow n \mid x - y$

$\exists z \in \mathbb{Z}$ $\quad x - y = n \cdot z$

$y - x = n \cdot (-z) \Rightarrow$

$\Rightarrow n \mid y - x$

$\Rightarrow y \sim_n x$

So $\sim_n$ is symmetric relation

3) T. Let $x, y, z \in \mathbb{Z}$: $\begin{cases} x \sim_n y \\ y \sim_n z \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} n \mid x-y \\ n \mid y-z \end{cases} \Leftarrow \begin{cases} \exists\, k_1 \in \mathbb{Z} & n \cdot k_1 = x-y \\ \exists\, k_2 \in \mathbb{Z} & n \cdot k_2 = y-z \end{cases} \Rightarrow$ (*)

$\Rightarrow n(k_1+k_2) = x-z \Rightarrow n \mid x-z$, so $x \sim_n y$, so (T)

o.k. a $x \equiv y \mod n$

$x \overset{n}{\equiv} y$

2) For $x \in \mathbb{Z}$; $\overline{x} = \{y \in \mathbb{Z} \mid y \sim_n x\}$

$n \mid y-x$

$\overline{x} = \{x + k \cdot n \mid k \in \mathbb{Z}\}$ $\exists k$ $y-x = k \cdot n$

3) $\overline{x} = \overline{y} \overset{\text{Prop}}{\Leftrightarrow} x \sim_n y \Leftrightarrow n \mid x-y$

System of representations:

$\hat{0} = \{0 + k \cdot n \mid k \in \mathbb{Z}\}$ → all multiples of n

$\hat{1} = \{1 + k \cdot n \mid k \in \mathbb{Z}\}$ → —— " —— $+1$

...

$\widehat{n-1}$

$\hat{n} = \{n + k \cdot n \mid k \in \mathbb{Z}\} = \hat{0}$

$\{0, 1, 2, \ldots, n-1\}$ is a system of representatives for $\sim_n$

Quotient space $\mathbb{Z}_n / \sim_n = \{\hat{x} \mid x \in \mathbb{Z}\} = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{n-1}\} = \mathbb{Z}_n$ notation

In fact $(\mathbb{Z}_n, +)$ is a ring)

$\hat{0}$ in $\mathbb{Z}_5$ is $\hat{0} = \{5k \mid k \in \mathbb{Z}\}$

$\hat{0}$ in $\mathbb{Z}_3$ is $\hat{0} = \{3k \mid k \in \mathbb{Z}\}$